Gaussian fields, equilibrium potentials and multiplicative chaos for Dirichlet forms *

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Abstract

For a Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(E;m)$, let $\mathbb{G}(\mathcal{E})=\{X_u;u\in\mathcal{F}_e\}$ be the Gaussian field indexed by the extended Dirichlet space $\mathcal{F}_e$. We first solve the equilibrium problem for a regular recurrent Dirichlet form $\mathcal{E}$ of finding for a closed set B a probability measure μ^B concentrated on B whose recurrent potential $R\mu^B\in\mathcal{F}_e$ is constant q.e. on B (called a Robin constant). We next assume that E is the complex plane $\mathbb{C}$ and $\mathcal{E}$ is a regular recurrent strongly local Dirichlet form. For the closed disk $\bar{B}(\mathbf{x},r)=\{\mathbf{z}\in\mathbb{C}:|\mathbf{z}-\mathbf{x}|\le r\}$, let $\mu^{\mathbf{x},r}$ and $f(\mathbf{x},r)$ be its equilibrium measure and Robin constant. Denote the Gaussian random variable $X_{R\mu^{\mathbf{x},r}}\in\mathbb{G}(\mathcal{E})$ by $Y^{\mathbf{x},r}$ and let, for a given constant $\gamma>0$, $\mu_r(A,\omega)=\int_A\exp(\gamma Y^{\mathbf{x},r}-(1/2)\gamma^2 f(\mathbf{x},r))d\mathbf{x}$. Under a certain condition on the growth rate of $f(\mathbf{x},r)$, we prove the convergence in probability of $\mu_r(A,\omega)$ to a random measure $\overline{\mu}(A,\omega)$ as $r\downarrow 0$. The possible range of γ to admit a non-trivial limit will then be examined in the cases that $(\mathcal{E}.\mathcal{F})$ equals $(\frac{1}{2}\mathbf{D}_{\mathbb{C}},H^1(\mathbb{C}))$ and $(\mathbf{a},H^1(\mathbb{C}))$, where $\mathbf{a}$ corresponds to the uniformly elliptic partial differential operator of divergence form.

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1 Introduction

Let E be a locally compact separable metric space, m a positive Radon measure on E with full support and $\mathcal{E}$ a regular Dirichlet form on $L^2(E;m)$. $(\mathcal{F}_e,\mathcal{E})$ denotes the extended Dirichlet space of $\mathcal{F}$. There are two stochastic objects associated with $\mathcal{E}$. One is an m-symmetric Markov process $\mathbb{M}=(\{X_t\}_{t\ge 0},\{\mathbb{P}_x\}_{x\in E})$ on E possessing nice properties called a Hunt process whose transition function $\{P_t;t>0\}$ generates the strongly continuous contraction semigroup on $L^2(E;m)$ associated with $\mathcal{E}$. Another is the centered Gaussian field $\mathbb{G}(\mathcal{E})=\{X_u;u\in\mathcal{F}_e\}$ indexed by $\mathcal{F}_e$ defined on a probability space $(\Omega,\mathcal{B},\mathbb{P})$ with covariance $\mathbb{E}[X_uX_v]=\mathcal{E}(u,v),\ u,v\in\mathcal{F}_e$. We like to study the structures and the properties of the Gaussian field $\mathbb{G}(\mathcal{E})$ by developing and using the probabilistic potential theory for the regular Dirichlet form $\mathcal{E}$ formulated in terms of the Hunt process $\mathbb{M}$.

Under the condition that $\mathcal{E}$ is transient, the potential theory for $\mathcal{E}$ as well as its probabilistic counterpart had been well developed by [BD, De, Si, F1] when M. Röckner [R] utilized this theory to establish the equivalence between the Markov property of the Gaussian field $\mathbb{G}(\mathcal{E})$ and the locality of the Dirichlet form $\mathcal{E}$. See Theorem 2.3 below and [MG]. In a recent paper [FO] by the present

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authors, such an equivalence is extended to irreducible recurrent Dirichlet forms $\mathcal{E}$ by making use of a newly introduced notion of recurrent potentials $R\mu \in \mathcal{F}_e$ of finite signed measures μ on E relative to an arbitrarily chosen admissible (compact) set $F \subset E$. In subsection 2.3 of the present paper, we shall also present an alternative proof of a part of [FO, Th.4.4] by means of reduction arguments to transient cases.

The primary purposes of the present paper are twofold. The first purpose is to develop in Section 3 the probabilistic potential theory of the regular Dirichlet form $\mathcal{E}$ further by solving the equilibrium problem for recurrent Dirichlet form $\mathcal{E}$ an electrostatic problem to find, for a set $B \subset E$, a probability measure μ^B concentrated on B whose potential $R\mu^B$ equals a constant (called the Robin constant) q.e. on B.

The second purpose concerns the special case that the underlying space E of the form $\mathcal{E}$ is the complex plane $\mathbb{C}$ or its subdomain, and we adopt in Section 4 the equilibrium measures μ^B and its potential $R\mu^B$ in constructing the Gaussian multiplicative chaos (GMC) a random measure on E created by exponentiating the Gaussian field $\mathbb{G}(\mathcal{E})$. Recently GMCs have been investigated intensively in the context of the Gaussian free field (GFF) related to mathematical physics under the name *Liouville* (*quantum gravity*) *measure* (cf. [K, DS, RV, Sh, B]).

The equilibrium problem for the logarithmic potential $U\mu(\mathbf{x}) = \frac{1}{\pi}\int_{\mathbb{C}} \log \frac{1}{|\mathbf{x}-\mathbf{y}|}\mu(d\mathbf{y}),\ \mathbf{x} \in \mathbb{C}$, on $\mathbb{C}$ was solved by De La Valée Poussin [VP, §2] for any non-polar bounded closed set $B \subset \mathbb{C}$ by finding a unique measure μ^B minimizing the logarithmic energy $\langle \mu, U\mu \rangle$ among all probability measures μ concentrated on B. Its probabilistic refinement was later presented in the book [PS] by S.C.Port and C.J.Stone published in 1978 along with the identification of μ^B with the hitting distribution of the planar Brownian motion $(X_t, \mathbb{P}_x)$ from infinity to B:

$$\mu^B(C) = \lim_{|\mathbf{x}|\to\infty} \mathbb{P}_{\mathbf{x}}(X_{\sigma_B} \in C), \quad \sigma_B = \inf\{t > 0 : X_t \in B\}, \quad C \in \mathcal{B}(\mathbb{C}).$$

When B is the closure of the open disk $B(r) = \{\mathbf{y} \in \mathbb{C} : |\mathbf{y}| < r\}$, μ^B is simply the uniform probability measure on $\partial B(r)$, while $U\mu^B(\mathbf{y}) = \frac{1}{\pi}\log\frac{1}{|\mathbf{y}|\vee r},\ \mathbf{y} \in \mathbb{C}$, so that the logarithmic energy $\langle \mu^B, U\mu^B \rangle = \frac{1}{\pi}\log\frac{1}{r}$ is negative for $r > 1$ and the Dirichlet integral of $U\mu^B$ diverges. In this sense, a direct use of the logarithmic potential of a positive finite measure is inconvenient for our purpose. See [F2, §5.2] and subsection 2.4 **(III)** below for one way out of such a trouble.

Since then, no substatial progress seems to have been made about the equilibrium problem for continuous time recurrent Markov processes except for the paper [O]. In Section 3, we incorporate the idea in this paper into a general setting of a regular recurrent Dirichlet form $\mathcal{E}$ on $L^2(E; m)$ under certain conditions on the resolvent of the associated Hunt process $\mathbb{M} = (X_t, \mathbb{P}_x)$ on E in the following manner. For an arbitrarily fixed admissible set $F \subset E$, let $\{R\mu : \mu \in \mathcal{M}_0\}$ be the family of recurrent potentials relative to F defined in §2.3. For any $A \in \mathcal{B}(E)$ with $m(A) > 0$, let B be the quasi-support of $1_A \cdot m$. ($B = \overline{A}$ whenever A is open and every point of ∂A is regular for A). Then

$$\mu^B(C) = \frac{1}{m(F)}\mathbb{P}_{1_F\cdot m}(X_{\sigma_B} \in C), \quad C \in \mathcal{B}(E)$$

is the unique probability measure in $\mathcal{M}_0$ concentrated on B such that its recurrent potential $R\mu^B$ relative to F takes a constant value $c(B) = m(F)^{-2}(1_F, R^{E\setminus B}1_F)_m$ q.e. on B, where $R^{E\setminus B}$ denotes the 0-order resolvent of the part of $\mathbb{M}$ on $E \setminus B$. Furthermore, μ^B is the unique measure minimizing $\mathcal{E}(R\mu, R\mu) = \langle \mu, R\mu \rangle$ among all probability measures $\mu \in \mathcal{M}_0$ concentrated on B and the minimum value equals the Robin constant $c(B)$.

When E is a bounded domain $D \subset \mathbb{C}$ and $(\mathcal{E}, \mathcal{F})$ is the transient Dirichlet form $(\frac{1}{2}\mathbf{D}_D, H_0^1(D))$ on $L^2(D)$ associated with the absorbing Brownian motion (ABM) on D, B. Duplantier and S. Sheffield

[DS] employed the uniform probability measure on the shrinking circle and the corresponding Gaussian random variable in $\mathbb{G}(\mathcal{E})$ to construct a Liouville random measure. Sheffield [Sh] also suggests analogous constructions in the cases of the recurrent Dirichlet forms $(\frac{1}{2}\mathbf{D}_{\mathbb{C}}, H^1(\mathbb{C}))$ and $(\frac{1}{2}\mathbf{D}_{\mathbb{H}}, H^1(\mathbb{H}))$ associated with the BM on $\mathbb{C}$ and the reflecting BM on the upper-half plane $\mathbb{H}$, respectively. In Section 4, we shall construct GMCs in more general planar cases by means of the equilibrium measure and the Robin constant in place of the uniform probability measure and the variance of the corresponding Gaussian random variable, respecively. There have been several methods used in constructing GMCs in transient cases. Among them, the method due to N. Berestycki [B] based on the Cameron-Martin formulae for the Gaussian field (see subsection 2.4) works in recurrent cases as well, and we shall invoke it in our construction.

More specifically, we consider in section 4 a regular recurrent strongly local Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathbb{C}, d\mathbf{x})$ with the associated diffusion $\mathbb{M}$ on $\mathbb{C}$ satisfying certain conditions including a Gaussian bound of the transition function. We fix an arbitrary $S > 2$, choose the annulus $F = \overline{B(S+1)} \setminus B(S)$ as an admissible set and consider the family $\{R\mu \in \mathcal{F}_e; \mu \in \mathcal{M}_0\}$ of recurrent potentials relative to F. For each disk $B(\mathbf{x}, \varepsilon) = \{\mathbf{y} \in \mathbb{C} : |\mathbf{y}-\mathbf{x}| < \varepsilon\}$ with $\mathbf{x} \in B(S-1)$, $\varepsilon \in (0, 1)$, denote by $\mu^{\mathbf{x},\varepsilon} \in \mathcal{M}_0$ and $f(\mathbf{x}, \varepsilon)$ the equilibrium measure and the Robin constant for the set $\overline{B(\mathbf{x}, \varepsilon)}$ relative to F, respectively.

Take any measure σ on $B(S-1)$ absolutely continuous with respect to the Lebesgue measure with a strictly positive bounded density. Let $\mathbb{G}(\mathcal{E}) = \{X_u : u \in \mathcal{F}_e\}$ be the centered Gaussian field defined on a probability space $(\Omega, \mathcal{B}, \mathbb{P})$ with covariance $\mathbb{E}[X_u X_v] = \mathcal{E}(u, v)$ and let $Y^{\mathbf{x},\varepsilon} = X_{R\mu^{\mathbf{x},\varepsilon}}$. For a fixed $\gamma > 0$, we put

$$\mu_\varepsilon(A, \omega) = \int_A \exp\left[\gamma Y^{\mathbf{x},\varepsilon} - \frac{\gamma^2}{2} f(\mathbf{x}, \varepsilon)\right] \sigma(d\mathbf{x}), \quad A \in \mathcal{B}(B(S-1)). \tag{1.1}$$

Under certain condition on the growth rate of the Robin constant $f(\mathbf{x}, \varepsilon)$ as $\varepsilon \downarrow 0$, we derive the convergence in probability of the random measure $\mu_\varepsilon(\cdot, \omega)$ as $\varepsilon \downarrow 0$ to a non-degenerate random measure $\overline{\mu}(\cdot, \omega)$ on $B(S-1)$ relative to a metric ρ on the space of all finite positive measures on $B(S-1)$ compatible with the weak convergence (Theorem 4.13). We call $\overline{\mu}(\cdot, \omega)$ the *Gaussian multiplicative chaos* (GMC) on $B(S-1)$ for the given Dirichlet form $(\mathcal{E}, \mathcal{F})$.

In section 5, we examine the possible range of the parameter $\gamma > 0$ to ensure the above mentioned convergence to a non-degenerate random measure in three examples where $(\mathcal{E}, \mathcal{F})$ equals $(\frac{1}{2}\mathbf{D}_{\mathbb{C}}, H^1(\mathbb{C}))$, $(\frac{1}{2}\mathbf{D}_{\mathbb{H}}, H^1(\mathbb{H}))$ and $(\mathbf{a}, H^1(\mathbb{C}))$. Here the form $\mathbf{a}$ is defined by

$$\mathbf{a}(u, v) = \sum_{i,j=1}^{2} \int_{\mathbb{C}} a_{ij}(\mathbf{x}) \frac{\partial u(\mathbf{x})}{\partial x_i} \frac{\partial u(\mathbf{x})}{\partial x_j} d\mathbf{x} \tag{1.2}$$

with measurable coefficients $a_{ij}(\mathbf{x})$, $\mathbf{x} \in \mathbb{C}$, satisfying

$$a_{ij}(\mathbf{x}) = a_{ji}(\mathbf{x}),\ 1 \le i, j \le 2, \quad \lambda|\xi|^2 \le \sum_{i,j=1}^{2} a_{ij}(\mathbf{x})\xi_i\xi_j \le \Lambda|\xi|^2,\ \xi \in \mathbb{R}^2,\ \mathbf{x} \in \mathbb{C}, \tag{1.3}$$

for some constants $0 < \lambda \le \Lambda$. In the first and second examples, the possible range of γ is shown to be equal to $(0, 2\sqrt{\pi})$. In the third example with $a_{ij} \in C^2(\mathbb{C})$, $1 \le i, j \le 2$, it is shown to be equal to $\left(0, 2\sqrt{\frac{2\pi\lambda^2\Lambda}{2\Lambda^2-\lambda^2}}\right)$, which reduces to $(0, 2\sqrt{\pi})$ when $a_{ij}(\mathbf{x}) = \frac{1}{2}\delta_{ij}$ as in the first example.

In subsecton 6.1, we consider a general regular transient strongly local Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(D, d\mathbf{x})$ for a domain $D \subset \mathbb{C}$ and make an analogous consideration to section 4 in constructing the associated GMC $\overline{\mu}(\cdot, \omega)$ on a bounded subdomain D_0 of D by means of a counterpart of (1.1).

The possible range of γ in the case that $(\mathcal{E}, \mathcal{F}) = (\frac{1}{2}\mathbf{D}_D, H_0^1(D))$ for a bounded domain $D \subset \mathbb{C}$ is also shown to be equal to $(0, 2\sqrt{\pi})$. In subsection 6.2, we further study in this case transformations of GMC by conformal maps of the domain D based on the conformal invariance of renormalized equilibrium potentials.

The systematic study of multiplicative chaos for Gaussian fields was initiated by J.-P. Kahane [K]. Specifically, given a kernel $K(x, y)$ with a logarithmic singularity on diagonal, the associated random measure was constructed in [K] using an approximation of K by sums of non-singular positive definite kernels, which is well applicable to massive GFF's. For massless GFF's, alternative approximations of K by its convolutions with mollifiers or measures of shrinking supports have been successfully utilized ([RV], [B]).

We start with a Dirichlet form $(\mathcal{E}, \mathcal{F})$ instead of a kernel K and construct the associated random measure by using directly the well defined equilibrium measures with shrinking supports. In transient cases like $(\frac{1}{2}\mathbf{D}_D, H_0^1(D))$ for a bounded domain $D \subset \mathbb{C}$, the Green function plays the role of the above mentioned kernel K, while, in recurrent cases like $(\frac{1}{2}\mathbf{D}_{\mathbb{C}}, H^1(\mathbb{C}))$, no Green function is available. But see [F2, §5.2] where yet another way of constructing a random measure is indicated.

We conjecture that the right endpoint of the possible range of γ examined in Section 5 and Exampe 6.3 is the critical value in the sense that the random measure $\overline{\mu}$ degenerates for that value of γ.

2 Basic properties of Gaussian field $\mathbb{G}(\mathcal{E})$

In this section, we discuss two basic properties of $\mathbb{G}(\mathcal{E})$; the Markov property and the Cameron-Martin formula. But the first one will not be used in the rest of this paper.

2.1 A pseudo Markov property of $\mathbb{G}(\mathcal{E})$

According to [Do] or [I], we have the following: given a set Λ equipped with $C(\lambda, \mu) \in \mathbb{R},\ \lambda, \mu \in \Lambda$, such that $C(\lambda, \mu) = C(\mu, \lambda)$ and $\{C(\lambda_i, \lambda_j)\}$ is non-negative definite for any finite $\{\lambda_i\} \subset \Lambda$, there exists uniquely Gaussian distributed random variables $\mathbb{G}(\Lambda) = \{X_\lambda; \lambda \in \Lambda\}$ defined on a probability space $(\Omega, \mathcal{B}, \mathbb{P})$ with

$$\mathbb{E}[X_\lambda \cdot X_\mu] = C(\lambda, \mu), \quad \mathbb{E}[X_\lambda] = 0, \quad \forall \lambda, \mu \in \Lambda,$$

whose finite linear combinations are Gaussian. $\mathbb{G}(\Lambda)$ is called the *Gaussian system* with index set Λ. When Λ is an Euclidean space $\mathbb{R}^d$ (resp. a function space), we may call $\mathbb{G}(\Lambda)$ a *Gaussian process* (resp. *Gaussian field*).

We recall that, in the study of the Markov property of Gaussian processes, the following useful notion and criterion were presented in H.P. McKean [M] and L.D. Pitt [P, Lem.2.1, Lem.2.2], respectively: for sub σ-algebras $\mathcal{F},\ \mathcal{G},\ \Sigma$ of $\mathcal{B}$, Σ is said to be a *splitting σ-algebra* for $\mathcal{F}$ and $\mathcal{G}$ if

$$\mathbb{P}(A \cap B|\Sigma) = \mathbb{P}(A|\Sigma) \cdot \mathbb{P}(B|\Sigma), \quad \forall A \in \mathcal{F}, \quad \forall B \in \mathcal{G}. \tag{2.1}$$

If $\mathcal{F} = \sigma(X_\lambda, \lambda \in \Lambda_1)$ and $\mathcal{G} = \sigma(X_\lambda, \lambda \in \Lambda_2)$ for $\Lambda_1,\ \Lambda_2 \subset \Lambda$ and if $\Sigma \subset \mathcal{F}$, then (2.1) is equivalent to the condition that

$$\sigma\left\{\mathbb{E}[X_\lambda \mid \mathcal{F}];\ \lambda \in \Lambda_2\right\} \subset \Sigma. \tag{2.2}$$

We may think of $\mathcal{F}$ (resp. $\mathcal{G}$) as the future (resp. past) events. As is well known, (2.1) is also equivalent to the condition that $\mathbb{P}(B \mid \mathcal{G}) = \mathbb{P}(B \mid \Sigma)$, for any $B \in \mathcal{F}$.

Throughout this paper, we are concerned with the Gaussian field with index set being a general extended Dirichlet space. Once for all, let E be a locally compact separable metric space, m an everywhere dense positive Radon measure on E and $(\mathcal{E}, \mathcal{F})$ a Dirichlet form on $L^2(E; m)$.

Let $\mathcal{F}_e$ be the collection of all m-measurable functions u on E such that $|u| < \infty$ m-a.e. and there exists an $\mathcal{E}$-Cauchy sequence $u_n \in \mathcal{F}$, $n \geq 1$, with $\lim_{n\to\infty} u_n = u$ m-a.e. $\mathcal{E}$ then extends from $\mathcal{F}$ to $\mathcal{F}_e$ as a non-negative symmetric bilinear form. $(\mathcal{F}_e, \mathcal{E})$ is called the *extended Dirichlet space of the Dirichlet form* $(\mathcal{E}, \mathcal{F})$ ([Si, FOT]). Let $\mathbb{G}(\mathcal{E}) = \{X_u : u \in \mathcal{F}_e\}$ be the Gaussian field defined on a probability space $(\Omega, \mathcal{B}, \mathbb{P})$ indexed by the functions of the space $\mathcal{F}_e$ and possessing the covariance $\mathbb{E}[X_u X_v] = \mathcal{E}(u, v),\ u, v \in \mathcal{F}_e$.

We now assume that the Dirichlet form $(\mathcal{E}.\mathcal{F})$ is regular. A function $u \in \mathcal{F}_e$ is called *$\mathcal{E}$-harmonic* on an open set $G \subset E$ if $\mathcal{E}(u, v) = 0$ for any $v \in \mathcal{F} \cap C_c(E)$ with $\text{supp}[v] \subset G$, where $C_c(E)$ is the family of continuous functions on E with compact support. Following A.Beurling and J.Deny [BD], the complement of the largest open set where u is harmonic will be called the *spectrum* of u and denoted by $s(u)$. See [De, p 166] and [FOT, p 99]. For any set $A \subset E$, we define the sub σ-algebra $\sigma(A)$ of $\mathcal{B}$ by

$$\sigma(A) = \sigma\{X_u : \quad u \in \mathcal{F}_e, \quad s(u) \subset A\}. \tag{2.3}$$

For any closed set $B \subset E$, let $\mathcal{F}_{e,E\setminus B}$ be a linear subspace of $\mathcal{F}_e$ defined by

$$\mathcal{F}_{e,E\setminus B} = \{u \in \mathcal{F}_e : \widetilde{u} = 0 \text{ q.e. on } B\}, \tag{2.4}$$

where $\widetilde{u}$ denotes a quasi-continuous version of u. By [FOT; Theorem 2.3.3], $s(u) \subset B$ if and only if

$$\mathcal{E}(u, v) = 0, \qquad \forall v \in \mathcal{F}_{e,E\setminus B}. \tag{2.5}$$

Let $\mathbb{M} = (X_t, \mathbb{P}_x)$ be the Hunt process on E associated with the regular Dirichlet form $(\mathcal{E}, \mathcal{F})$. $\mathcal{B}(E)$ will denote the totality of Borel subsets of E. For any $B \in \mathcal{B}(E)$, the *hitting distribution* $H_B(x, \cdot)$ of $\mathbb{M} = (X_t, \mathbb{P}_x)$ for B is defined by $H_B f(x) = \mathbb{E}_x[f(X_{\sigma_B})]$, $x \in E$, for any bounded Borel functions f on E where $\sigma_B = \inf\{t > 0 : X_t \in B\}$. In view of [FOT, Th.4.6.5] or [CF, Th.3.4.8], it holds for any closed set B and any $u \in \mathcal{F}_e$ that $H_B|\widetilde{u}|(x) < \infty$ for q.e. $x \in E$ and $H_B\widetilde{u}$ is a quasi-continuous element of $\mathcal{F}_e$ satisfying (2.5). Hence

$$s(H_B\widetilde{u}) \ \subset \ B, \text{ for any closed set } B \subset E \text{ and for any } u \in \mathcal{F}_e. \tag{2.6}$$

Lemma 2.1 *The Gaussian field* $\mathbb{G}(\mathcal{E}) = \{X_u : u \in \mathcal{F}_e\}$ *enjoys the following property: For any closed set* $B \subset E$ *and any* $u \in \mathcal{F}_e$,

$$X_{u - H_B\widetilde{u}} \quad \text{is independent of} \quad \sigma(B), \tag{2.7}$$

and

$$\mathbb{E}[X_u \mid \sigma(B)] = X_{H_B\widetilde{u}}. \tag{2.8}$$

Proof. Take any $v \in \mathcal{F}_e$ with $s(v) \subset B$. Since $u - H_B\tilde{u} \in \mathcal{F}_{e,E\setminus B}$, $\mathcal{E}(u - H_B\tilde{u}, v) = 0$ by (2.5). Hence $\mathbb{E}\left[(X_u - X_{H_B\tilde{u}})X_v\right] = 0$ so that (2.7) holds as all random variables involved are centered Gaussian. Consequently $\mathbb{E}\left[X_u - X_{H_B\widetilde{u}} | \sigma(B)\right] = \mathbb{E}\left[X_u - X_{H_B\widetilde{u}}\right] = 0$, and so (2.8) is valid by (2.6). □

(2.8) is a *fundamental identity of the Gaussian field* $\mathbb{G}(\mathcal{E})$. It follows from (2.8) and the criterion (2.2) that, for any set $A \subset E$,

$$\sigma\{X_{H_{\overline{A}}\widetilde{u}} : u \in \mathcal{F}_e,\ s(u) \subset \overline{E \setminus A}\} \text{ is a splitting } \sigma\text{-algebra for } \sigma(\overline{E \setminus A}) \text{ and } \sigma(\overline{A}). \tag{2.9}$$

We may call (2.9) a *pseudo Markov property* of the Gaussian field $\mathbb{G}(\mathcal{E})$.

The Gaussian field $\mathbb{G}(\mathcal{E})$ is said to possess the *Markov property with respect to a set* $A \subset E$ if

$$\sigma(\partial A) \text{ is a splitting } \sigma\text{-algebra for } \sigma(\overline{E \setminus A}) \text{ and } \sigma(\overline{A}). \tag{2.10}$$

We say that $\mathbb{G}(\mathcal{E})$ has the *Markov property* if it possesses the Markov property with respect to any subset A of E.

2.2 Characterization of Markov property of $\mathbb{G}(\mathcal{E})$ for transient $\mathcal{E}$

Let us assume that the regular Dirichlet form $\mathcal{E}$ is transient, or equivalently, that there exists a bounded m-integrable function h strictly positive m-a.e. on E sstisfying

$$(|u|, h) \le \sqrt{\mathcal{E}(u,u)} \quad \text{for any } u \in \mathcal{F}. \tag{2.11}$$

This inequality is extended to any $u \in \mathcal{F}_e$ and $\mathcal{F}_e$ becomes a real Hilbert space with inner product $\mathcal{E}$. The function h in (2.11) is called a *reference function* (cf. [FOT, Th.1.5.1]).

A positive Radon measure μ on E is called a *measure of finite* 0*-order energy* and we write as $\mu \in \mathcal{S}_0^{(0)}$ if there exists a positive constant C such that

$$\langle |u|, \mu\rangle \le C\sqrt{\mathcal{E}(u,u)} \quad \text{for all } u \in \mathcal{F} \cap C_c(E). \tag{2.12}$$

We let $\mathcal{M}_0 = \{\mu = \nu_1 - \nu_2 : \nu_i \in \mathcal{S}_0^{(0)},\ i = 1,2\}$. Any $\mu \in \mathcal{M}_0$ then admits a unique function $U\mu \in \mathcal{F}_e$ satisfying the *Poisson equation*

$$\mathcal{E}(U\mu, u) = \langle \mu, \widetilde{u}\rangle \quad \text{for any } u \in \mathcal{F}_e. \tag{2.13}$$

$U\mu$ is called the *potential of the measure* $\mu \in \mathcal{M}_0$. For a Borel set $B \subset E$ and a signed Radon meausre μ on E, define

$$\mu_B(A) = \int_E \mu(dx) H_B(x, A), \quad \forall A \in \mathcal{B}(E). \tag{2.14}$$

Lemma 2.2 (i) *For any closed set $B \subset E$ and for any $\mu \in \mathcal{M}_0$, $\mu_B \in \mathcal{M}_0$ and*

$$H_B(\widetilde{U\mu}) \;=\; U\mu_B. \tag{2.15}$$

(ii) $s(U\mu) = \mathrm{supp}(|\mu|)$ *for* $\mu \in \mathcal{M}_0$.
(iii) (Spectral synthesis) *For any $u \in \mathcal{F}_e$, there exists a sequence $\mu_n \in \mathcal{M}_0,\ n \ge 1$, such that* $\mathrm{supp}[|\mu_n|] \subset s(u),\ n \ge 1$, *and $U\mu_n$ is $\mathcal{E}$-convergent to u.*

Proof. (i). The map $u \mapsto H_B\widetilde{u}$ defines a projection from $\mathcal{F}_e$ to the orthogonal complement of $\mathcal{F}_{e,E\setminus B}$. Hence

$$\langle \mu_B, v\rangle = \langle \mu, H_B v\rangle = \mathcal{E}(U\mu, H_B v) = \mathcal{E}(H_B\widetilde{U\mu}, v), \quad \mu \in \mathcal{M}_0,\ v \in \mathcal{F} \cap C_c(E),$$

and $\langle \mu_B, |v|\rangle \le \sqrt{\langle \mu, \widetilde{U\mu}\rangle}\,\sqrt{\mathcal{E}(v,v)},\ \mu \in \mathcal{S}_0^{(0)},\ v \in \mathcal{F} \cap C_c(E)$, from which the assertions follow.
(ii). The space $C(S)$ of continuous functions on a locally compact Hausdorff space S vanishing at infinity admits as its dual space the space of finite signed measures on S normed by their total variations. (ii) follows from this and the Poisson equation (2.13) holding for $u \in \mathcal{F}_e \cap C_c(E)$ with $\mathrm{supp}[u] \subset E_0$ for each relatively compact open set $E_0 \subset E$. .
(iii). Since $vh \cdot m \in \mathcal{M}_0$ for the reference function h and for any $v \in C_c(E)$, (2.13) also implies that, for any $u \in \mathcal{F}_e$, there exists a sequence $\{U\nu_n : \nu_n \in \mathcal{M}_0\}$ which is $\mathcal{E}$-convergent to u. Let $\mu_n = (\nu_n)_B,\ n \ge 1$, for $B = s(u)$. Then $s(U\mu_n) \subset B$ by (ii) and $\{U\mu_n\}$ is $\mathcal{E}$-convergent to $H_B\widetilde{u}$ as $n \to \infty$ by (2.15). Since $\mathcal{E}(u - H_B\widetilde{u}, v) = 0$ for any $v \in \mathcal{F}_{e,E\setminus B}$ by (2.5) and (2.6) and $u - H_B\widetilde{u} \in \mathcal{F}_{e,E\setminus B}$, we get $H_B\widetilde{u} = u$ by the transience of $\mathcal{E}$. □

Lemma 2.2 (iii) is a 0-order version of [FOT, Th. 2.3.2]. We like to take this opportunity to mention that the phrase '$\mathcal{E}_1$-convergent limit' on the 6-th line in the proof of this theorem of [FOT] is better to be replaced by '$\mathcal{E}_\alpha$-weakly convergent limit'. A regular Dirichlet form $\mathcal{E}$ is said to be *local* if $\mathcal{E}(u,v) = 0$ whenever $u, v \in \mathcal{F}$ and $\mathrm{supp}[u \cdot m]$ and $\mathrm{supp}[v \cdot m]$ are disjoint compact set. The following theorem was established by M. Röckner [R]. We give its straightforward proof for completeness.

Theorem 2.3 *Suppose $\mathcal{E}$ is transient. Then the Gaussian field $\mathbb{G}(\mathcal{E})$ enjoys the Markov property if and only if the form $\mathcal{E}$ is local.*

Proof. In view of (2.2) and (2.8), $\mathbb{G}(\mathcal{E})$ has the Markov property if and only if, for any $A \subset E$,

$$\sigma(X_{H_{\overline{A}}\widetilde{u}} : u \in \mathcal{F}_e,\ s(u) \subset \overline{E \setminus A}) \ \subset \ \sigma(\partial A). \tag{2.16}$$

Assume that $\mathcal{E}$ is local. Then the Hunt process $\mathbb{M}$ associated with $\mathcal{E}$ is of continuous sample paths (cf. [FOT, Th.4.5.1]). Therefore the balayage $\mu_{\overline{A}}$ of $\mu \in \mathcal{M}_0$ with $\mathrm{supp}(|\mu|) \subset \overline{E \setminus A}$ has the support concentrated on ∂A so that, by Lemma 2.2 (i), (ii), $s(H_{\overline{A}}U\mu) = \mathrm{supp}(|\mu_{\overline{A}}|) \subset \partial A$.

On account of the spectral synthesis Lemma 2.2 (iii), there exists for any $u \in \mathcal{F}_e$ with $s(u) \subset \overline{E \setminus A}$ a sequence $\mu_n \in \mathcal{M}_0$, $n \geq 1$, such that $\mathrm{supp}[|\mu_n|] \subset \overline{E \setminus A}$, $n \geq 1$, and $U\mu_n$ is $\mathcal{E}$-convergent to u. Then $s(H_{\overline{A}}U\mu_n) \subset \partial A$ and $H_{\overline{A}}U\mu_n$ is $\mathcal{E}$-convergent to $H_{\overline{A}}u$ as $n \to \infty$ so that $s(H_{\overline{A}}u) \subset \partial A$, yielding the Markov property (2.16) of $\mathbb{G}(\mathcal{E})$.

Conversely assume that $\mathbb{G}(\mathcal{E})$ satisfies the Markov property (2.16). Let $G \subset E$ be an open set and u be any function in $\mathcal{F}_e$ with $s(u) \subset E \setminus G$. Then $X_{H_{\overline{G}}\widetilde{u}} \in \sigma(\partial G)$. Now take any open subset A of G with $\overline{A} \subset G$ and let $B = G \setminus \overline{A}$. Then $\sigma(\overline{G}) \supset \sigma(\overline{B}) \supset \sigma(\partial G)$. As $X_{H_{\overline{G}}\widetilde{u}} = \mathbb{E}[X_u | \sigma(\overline{G})]$ by (2.8), this means that $X_{H_{\overline{G}}\widetilde{u}} = \mathbb{E}[X_u | \sigma(\overline{B})] = X_{H_{\overline{B}}\widetilde{u}}$ and hence

$$\mathbb{E}[(X_{H_{\overline{G}}\widetilde{u}} - X_{H_{\overline{B}}\widetilde{u}})^2] = 0, \quad \text{that is,} \quad \mathcal{E}\left(H_{\overline{G}}\widetilde{u} - H_{\overline{B}}\widetilde{u}, H_{\overline{G}}\widetilde{u} - H_{\overline{B}}\widetilde{u}\right) = 0.$$

and so $H_{\overline{G}}\widetilde{u} = H_{\overline{B}}\widetilde{u}$.

By virtue of Lemma 2.2 (i), (ii) and the equation (2.13), we have, for any $\mu \in \mathcal{M}_0$ with $\mathrm{supp}(|\mu|) \subset E \setminus G$, $\langle \mu_{\overline{G}}, f \rangle = \langle \mu_{\overline{B}}, f \rangle$ for any $f \in \mathcal{F} \cap C_c(E)$. In particular, if the support of f is contained in A, then $\langle \mu_{\overline{G}}, f \rangle = 0$ so that

$$\langle \mu, H_{\overline{G}}f \rangle = 0 \quad \text{for any } \mu \in \mathcal{M}_0 \text{ with } \mathrm{supp}[|\mu|] \subset E \setminus \overline{G}.$$

This implies that $H_{\overline{G}}f = 0$ q.e. on $E \setminus \overline{G}$, and consequently $H_{\overline{G}}(x, \cdot)$ is concentrated on $\overline{B}$ for q.e. $x \in E \setminus \overline{G}$. Since this holds for any open subset A of G such that $\overline{A} \subset G$, $H_{\overline{G}}(x, \cdot)$ is concentrated on ∂G for q.e. $x \in E \setminus G$. Then the local property of $\mathcal{E}$ follows from [FOT, Lem.4.5.1]. □

2.3 Characterization of Markov property of $\mathbb{G}(\mathcal{E})$ for recurrent $\mathcal{E}$

Let us assume that the regular Dirichlet form $\mathcal{E}$ is irreducible recurrent. In particular, the constant function 1 is in $\mathcal{F}_e$ and $\mathcal{E}(1, 1) = 0$.

We make an additional assumption that the transition function $\{P_t; t > 0\}$ of the associated Hunt process $\mathbb{M}$ satisfies the following *absolute continuity condition*:

(AC) there exists a certain Borel properly exceptional set $N \subset E$ such that

$P_t(x, \cdot)$ is absolutely continuous with resepct to m for each $t > 0$ and $x \in E \setminus N$,

This condition is much milder than the one admitting no exceptional set N. For instance, it is fulfilled when the form $\mathcal{E}$ satisfies a Sobolev type inequality (cf. [FOT, Th.2.7]).

Under the assumption **(AC)**, the resolvent kernel $\{R_\alpha, \alpha > 0\}$ of $\mathbb{M}$ admits a density function $r_\alpha(x, y),\ x, y \in E \setminus N$, with respect to m such that it is strictly positive, symmetric Borel measurable, α-excessive relative to $\mathbb{M}$ in each variable, and it satisfies the resolvent equation. A set $F \subset E \setminus N$ is called an *admissible set* if

$$\begin{cases} F \text{ is compact, } m(F) > 0 \quad \text{and for some} \quad c > 0 \text{ and } \frac{1}{2} < a < 1, \\ m(\{y \in F : r_1(x, y) > c\}) > am(F) \quad \text{for every} \quad x \in F. \end{cases} \tag{2.17}$$

It has been shown by [FO, Lem.3.1] that, for any Borel set $B \subset E \setminus N$ with $m(B) > 0$, there exists an admissible set F contained in B. For a fixed admissible set F, we denote its indicator function 1_F by g and consider the perturbed form

$$\mathcal{E}^g(u,v) = \mathcal{E}(u,v) + \int_E uvgdm, \quad u,v \in \mathcal{F}^g = \mathcal{F} \cap L^2(E; g \cdot m),$$

which is a regular transient Dirichlet form on $L^2(E;m)$. Its extended Dirichlet space $\mathcal{F}_e^g$ equals $\mathcal{F}_e \cap L^2(E; g \cdot m)$. Let $\mathcal{S}_0^{g,(0)}$ be the space of positive Radon measures on E with finite 0-order energy relative to the form $\mathcal{E}^g$. Define

$$\mathcal{M}_0 = \{\mu = \mu_1 - \mu_2 : \mu_i \in S_0^{g,(0)}, \mu_i(E) < \infty, i = 1,2\}, \quad \mathcal{M}_{00} = \{\mu \in \mathcal{M}_0 : \mu(E) = 0\}. \tag{2.18}$$

By virtue of [FO, Th.3.5], there exists for any $\mu \in \mathcal{M}_0$ a quasi continuous function $R\mu \in \mathcal{F}_e^g$ uniquely up to q.e. equivalence such that

$$\mathcal{E}(R\mu, u) = \left\langle \mu, \widetilde{u} - \frac{1}{m(F)} \langle g \cdot m, u \rangle \right\rangle \quad \forall u \in \mathcal{F}_e^g, \quad \text{and} \quad \langle g \cdot m, R\mu \rangle = 0. \tag{2.19}$$

The first equation in the above determines $R\mu \in \mathcal{F}_e^g$ up to an additive constant (see **(H)** below), while the second identity is its normalization. In particular, we have the symmetry

$$\langle \mu, R\nu \rangle = \langle R\mu, \nu \rangle (= \mathcal{E}(R\mu, R\nu)), \quad \mu, \nu \in \mathcal{M}_0. \tag{2.20}$$

We call $\{R\mu : \mu \in \mathcal{M}_0\}$ the *family of recurrent potentials relative to an admissible set* F.

Contrarily to the transient case, the class $\mathcal{M}_0$ of measures and potentials $R\mu$, $\mu \in \mathcal{M}_0$, depend on the choice of an admissible set F, making relevent arguments more involved.
But the first equation in (2.19) has enabled us to derive in [FOT, Cor.4.8.2] and [FO, Th.3.7] a nice property of $\mathcal{E}$ that the quotient space $\dot{\mathcal{F}}_e$ of $\mathcal{F}_e$ by the space of constant functions is a real Hilbert space with inner product $\mathcal{E}$.

Theorem 2.4 (1) *Suppose that $\mathcal{E}$ is an irreducible recurrent regular Dirichlet form satisfying the condition* **(AC)**. *If $\mathcal{E}$ is local, then $\mathbb{G}(\mathcal{E})$ has the Markov property with respect to any open set.*
(2) *Suppose that $\mathcal{E}$ is an irreducible recurrent regular Dirichlet form. If $\mathbb{G}(\mathcal{E})$ has the Markov property with respect to any open set, then $\mathcal{E}$ is local.*

Remark 2.5 The first statement (1) of Theorem 2.4 has been proved in [FO, Th.4.4] by using some detailed properties of recurrent potentials $\{R\mu\}$ specified in the above.

The second statement (2) of Theorem 2.4 has been asserted also in [FO, Th.4.4] under the assumption **(AC)** for $\mathcal{E}$. We now give its proof without assuming **(AC)** by using a reduction argument to Theorem 2.3 for transient cases.

We would like to take this opportunity to mention that the proof of the implication (ii) ⇒ (iii) of [FO, Th.4.4] contains a flaw: the space $\mathcal{M}_0$ of measures there should be replaced by $\mathcal{M}_{00}$ and hence that proof works only under the additional condition that 'for any open set $G \subset E$ with $m(E \setminus G) > 0$, ∂G is of positive capacity'.

Theorem 2.4 (2) will be proved by using the following lemma.

Assume that the regular Dirichlet form $\mathcal{E}$ is irreducnible recurrent. Let E_0 be any open subset of E with $m(E \setminus E_0) > 0$ and $\mathcal{E}^{(0)}$ be the part of $\mathcal{E}$ on the set E_0.

The Gaussian field $\mathbb{G}(\mathcal{E}^{(0)})$ associated with $\mathcal{E}^{(0)}$ is the sub-field of $\mathbb{G}(\mathcal{E})$ obtained just by the restriction of the index set $\mathcal{F}_e$ to $\mathcal{F}_{e,E_0}$.

Lemma 2.6 *If $\mathbb{G}(\mathcal{E})$ has the Markov property with respect to any open set, then so does $\mathbb{G}(\mathcal{E}^{(0)})$.*

Proof. Put $E \setminus E_0 = B_0$. For $A \subset E_0$, define

$$\sigma(A) = \{X_u : u \in \mathcal{F}_e,\ s(u) \subset A\}, \quad \sigma^{(0)}(A) = \{X_u : u \in \mathcal{F}_{e,E_0},\ s^{(0)}(u) \subset A\}.$$

Take any open set $G \subset E_0$ with $B = \overline{G} \subset E_0$ and any $u \in \mathcal{F}_{e,E_0}$ with $s^{(0)}(u) \subset E_0 \setminus G$. By (2.8), $\mathbb{E}[X_u \big| \sigma(B \cup B_0)] = X_{H_{B\cup B_0}\widetilde{u}}$.

Define $H_B^{(0)} f(x) = \mathbb{E}_x[f(X_{\sigma_B}); \sigma_B < \sigma_{B_0}]$. Since $\widetilde{u} = 0$ q.e. on B_0,

$$H_{B\cup B_0}\widetilde{u}(x) = \mathbb{E}_x\left[\widetilde{u}(X_{\sigma_{B\cup B_0}})\right] = \mathbb{E}_x\left[\widetilde{u}(X_{\sigma_B \wedge \sigma_{B_0}})\right] = H_B^{(0)}\widetilde{u}(x),$$

and so

$$\mathbb{E}[X_u \big| \sigma(B \cup B_0)] = X_{H_B^{(0)}\widetilde{u}}. \tag{2.21}$$

As $s^{(0)}(H_B^{(0)}\widetilde{u}) \subset B$, $X_{H_B^{(0)}\widetilde{u}} \in \sigma^{(0)}(B)$. Hence we get from (2.21),

$$\mathbb{E}[X_u \big| \sigma^{(0)}(B)] = X_{H_B^{(0)}\widetilde{u}}. \tag{2.22}$$

By the Markov property of $\mathbb{G}(\mathcal{E})$ and (2.21), $X_{H_B^{(0)}\widetilde{u}} \in \sigma(\partial B \cup \partial B_0)$. This means that $s(H_B^{(0)}\widetilde{u}) \subset \partial B \cup \partial B_0$, namely, $\mathcal{E}(H_B^{(0)}\widetilde{u}, \varphi) = 0$ for any $\varphi \in \mathcal{F} \cap C_c(E)$ with $\mathrm{supp}[\varphi] \subset E \setminus (\partial B \cup \partial B_0)$. In particular, $s^{(0)}(H_B^{(0)}\widetilde{u}) \subset \partial B$ so that $X_{H_B^{(0)}\widetilde{u}} \subset \sigma^{(0)}(\partial B)$.

Therefore, (2.22) implies the Markov property of $\mathbb{G}(\mathcal{E}^0)$ relative to G for the part $\mathcal{E}^0$ of $\mathcal{E}$ on E_0. □

Proof of Theorem 2.4 (2). Assume that $\mathbb{G}(\mathcal{E})$ has the Markov property with respect to any open set. For any open set $E_0 \subset E$ with $m(E \setminus E_0) > 0$, the part $\mathcal{E}^{(0)}$ of $\mathcal{E}$ on E_0 is a transient Dirichlet form and the associated Gaussian field $\mathbb{G}(\mathcal{E}^{(0)})$ enjoys the Markov property with resepct to any open subset of E_0 by Lemma 2.7. By the proof of the 'only if' part of Theorem 2.3, the form $\mathcal{E}^{(0)}$ is local.

Take two open sets $E_0^i \subset E,\ i = 1, 2$, such that, $B_i = E \setminus E_0^i,\ i = 1, 2$, are compact, of positive m measure and mutually disjoint. Choose $\varepsilon > 0$ in such a way that the closures of ε-neighborhoods $B_{i,\varepsilon}$ of B_i are disjoint. Since the parts of the form $\mathcal{E}$ on E_0^1 and E_0^2 are local, we see from [FOT, Lem.4.5.1, Th.4.5.1] that the sample path X_t of the Hunt process $\mathbb{M} = (X_t, \mathbb{P}_x)$ associated with $\mathcal{E}$ is almost surely continuous on $[0, \sigma]$ where $\sigma = \sigma_{B_{1,\varepsilon}} \vee \sigma_{B_{2,\varepsilon}} \in (0, \infty)$.

Define $\sigma_0 = 0, \quad \sigma_1 = \sigma, \quad \sigma_n = \sigma_{n-1} + \sigma \circ \theta_{\sigma_{n-1}}, \quad n \geq 1$. Then

$$\mathbb{P}_x(X_t \text{ is not continuous on } [0, \sigma_n]) \leq \sum_{k=0}^{n-1} \mathbb{P}_x(X_t \text{ is not continuous on } [\sigma_k, \sigma_{k+1}])$$
$$= \sum_{k=0}^{n-1} \mathbb{E}_x\left[\mathbb{P}_{X_{\sigma_k}}(X_t \text{ is not continuous on } [0, \sigma])\right] = 0, \quad x \in E,$$

Hence the sample path of $\mathbb{M}$ is continuous a.s. on $[0, \widehat{\sigma})$ where $\widehat{\sigma} = \lim_{n\to\infty} \sigma_n$.

Suppose $\widehat{\sigma} < \infty$, then $\sigma \circ \widehat{\sigma} = 0$. On the other hand, due to the quasi-left continuity of the Hunt process $\mathbb{M}$, $X_{\widehat{\sigma}} \in \overline{B_{1,\varepsilon}} \cup \overline{B_{2,\varepsilon}}$ and so $\sigma \circ \widehat{\sigma} > 0$, a contradiction. Therefore $\mathbb{M}$ is a diffusion and hence $\mathcal{E}$ is local. □

2.4 Cameron-Martin formulae for $\mathbb{G}(\mathcal{E})$

Theorem 2.7 *Let $(\mathcal{E},\mathcal{F})$ be a general (not necesarily regular) Dirichlet form on $L^2(E;m)$ with the extended Dirichlet space $\mathcal{F}_e$ and let $\mathbb{G}(\mathcal{E})=\{X_u; u\in\mathcal{F}_e\}$ be the Gaussian field defined on a probability space $(\Omega,\mathcal{B},\mathbb{P})$ with covariance $\mathbb{E}[X_uX_v]=\mathcal{E}(u,v),\ u,v\in\mathcal{F}_e$. Then, for any $v_1,v_2,\cdots,v_n\in\mathcal{F}_e$, any $u\in\mathcal{F}_e$ and any bounded Borel function H on $\mathbb{R}^n$,*

$$\mathbb{E}\left[H(X_{v_1}+\mathcal{E}(u,v_1),\cdots,X_{v_n}+\mathcal{E}(u,v_n))\right]=e^{-\frac{1}{2}\mathcal{E}(u,u)}\mathbb{E}\left[e^{X_u}H(X_{v_1},\cdots,X_{v_n})\right]. \tag{2.23}$$

This *Cameron-Martin formula* can be readily proved by using the characteristic function. Another simple proof is being provided in the first half of "Alternative proof of Theorem 11.4.1" of M.B.Marcus and J. Rosen [MR, p.518], which indeed works by assuming that u equals one of $\{v_i, 1\le i\le n\}$ (otherwise it suffices to add a new index $v_{n+1}=u$).

We now assume that $\mathcal{E}$ is regular and derive from (2.23) those identities formulated in terms of signed measures on E of finite energy in four cases separately.

(I) Transient case. In view of the Poisson equation (2.13), the map $\nu\in\mathcal{M}_0\mapsto U\nu\in\mathcal{F}_e$ is injective. For $\nu\in\mathcal{M}_0$, we write $X_{U\nu}$ as Z_ν and regard $\{Z_\nu:\nu\in\mathcal{M}_0\}$ as a Gaussian field indexed by $\mathcal{M}_0$ with covariance $\langle\mu,\widetilde{U\nu}\rangle,\ \mu,\nu\in\mathcal{M}_0$. $\{Z_\nu:\nu\in\mathcal{M}_0\}$ can be thus identified with a subfield of $\mathbb{G}(\mathcal{E})$. The formula (2.23) is then rewritten as follows:
For any $\nu_1,\nu_2,\cdots,\nu_n\in\mathcal{M}_0$, any $u\in\mathcal{F}_e$ and any bounded Borel function H on $\mathbb{R}^n$

$$\mathbb{E}\left[H(Z_{\nu_1}+\langle\widetilde{u},\nu_1\rangle,\cdots,Z_{\nu_n}+\langle\widetilde{u},\nu_n\rangle)\right]=e^{-\frac{1}{2}\mathcal{E}(u,u)}\mathbb{E}\left[e^{X_u}H(Z_{\nu_1},\cdots,Z_{\nu_n})\right]. \tag{2.24}$$

(II) Irreducible recurrent case fulfilling condition (AC). In this case, choose any admissible set F and consider the family $\{R\mu:\mu\in\mathcal{M}_0\}$ of recurrent potentials relative to F. The spaces $\mathcal{M}_0$ and $\mathcal{M}_{00}$ of measures are defined by (2.18). We write $X_{R\nu}$ as Z_ν for $\nu\in\mathcal{M}_0$, In view of the generalized Poisson equation (2.19), the map $\mu\in\mathcal{M}_{00}\mapsto R\mu\in\mathcal{F}_e^g\subset\mathcal{F}_e$ is injective so that $\{Z_\nu:\nu\in\mathcal{M}_{00}\}$ can be identified with a Gaussian sub-field of $\mathbb{G}(\mathcal{E})$ and (2.24) holds true for $\mathcal{M}_{00}$ in place of $\mathcal{M}_0$.

The map $\mu\in\mathcal{M}_0\mapsto R\mu\in\mathcal{F}_e^g\subset\mathcal{F}_e$ is not injective. Nevertheless, we have from (2.20) and (2.23) the following formula similar to (2.24): For any $\nu_1,\nu_2,\cdots,\nu_n\in\mathcal{M}_0$, any $\mu\in\mathcal{M}_0$ and any bounded Borel function H on $\mathbb{R}^n$

$$\mathbb{E}\left[H(Z_{\nu_1}+\langle\mu,R\nu_1\rangle,\cdots,Z_{\nu_n}+\langle\mu,R\nu_n\rangle)\right]=e^{-\frac{1}{2}\langle\mu,R\mu\rangle}\mathbb{E}\left[e^{Z_\mu}H(Z_{\nu_1},\cdots,Z_{\nu_n})\right]. \tag{2.25}$$

(III) The case that $(\mathcal{E},\mathcal{F})=(\frac{1}{2}\mathbf{D}_{\mathbb{C}},H^1(\mathbb{C}))$. This is the Dirichlet form on $L^2(\mathbb{C})$ associated with the planar Brownian motion. $\mathbf{D}_{\mathbb{C}}$ is the Dirichlet integral on $\mathbb{C}$. $\mathcal{F}_e$ is the *Beppo Levi space* $\mathrm{BL}(\mathbb{C})=\{u\in L^2_{\rm loc}(\mathbb{C}):|\nabla u|\in L^2(\mathbb{C})\}$. $\mathbb{G}(\mathcal{E})$ satisfies the formula (2.23) for this choice of $(\mathcal{F}_e,\mathcal{E})$.

Let $\overset{\circ}{\mathcal{M}}_0(\mathbb{C})$ be the space of compactly supported finite signed measures on $\mathbb{C}$ with finite logarithmic energy and let $\overset{\circ}{\mathcal{M}}_{00}(\mathbb{C})=\{\mu\in\overset{\circ}{\mathcal{M}}_0(\mathbb{C}):\mu(\mathbb{C})=0\}$. The logarithmic potential $U\mu$ of $\mu\in\overset{\circ}{\mathcal{M}}_0(\mathbb{C})$ is defined by $U\mu(\mathbf{x})=\dfrac{1}{\pi}\displaystyle\int_{\mathbb{C}}\log\frac{1}{|\mathbf{x}-\mathbf{y}|}\mu(d\mathbf{y})$. By the Poisson equation in [F2, Th.2.6], $\mu\mapsto U\mu$ defines an injective map from $\overset{\circ}{\mathcal{M}}_{00}(\mathbb{C})$ into $\mathrm{BL}(\mathbb{C})$. For $\mu\in\overset{\circ}{\mathcal{M}}_{00}(\mathbb{C})$, we write $X_{U\mu}\in\mathbb{G}(\mathcal{E})$ as Z_μ. Then $\{Z_\mu:\mu\in\overset{\circ}{\mathcal{M}}_{00}(\mathbb{C})\}$ is a Gaussian field indexed by $\overset{\circ}{\mathcal{M}}_{00}(\mathbb{C})$ with covariance $\langle\mu,\widetilde{U\nu}\rangle,\ \mu,\nu\in\overset{\circ}{\mathcal{M}}_{00}(\mathbb{C})$. This field is designated in [F2] as $\mathbb{G}(\mathbb{C})$, which can be identified with a sub-field of $\mathbb{G}(\mathcal{E})$. The Cameron-Martin formula (2.24) holds for $\overset{\circ}{\mathcal{M}}_{00}(\mathbb{C})$, $\mathrm{BL}(\mathbb{C})$ and $\frac{1}{2}\mathbf{D}_{\mathbb{C}}$ in place of $\mathcal{M}_0$, $\mathcal{F}_e$ and $\mathcal{E}$, respectively.

(IV) The case that $(\mathcal{E},\mathcal{F})=(\frac{1}{2}\mathbf{D}_{\mathbb{R}},H^1(\mathbb{R}))$. This is the Dirichlet form on $L^2(\mathbb{R})$ associated with the standard Brownian motion on $\mathbb{R}$. $\mathbf{D}_{\mathbb{R}}$ is the Dirichlet integral on $\mathbb{R}$. $\mathcal{F}_e$ is the *Cameron-Martin space* $H^1_e(\mathbb{R})=\{u:$ absolutely continuous on $\mathbb{R},\ \mathbf{D}_{\mathbb{R}}(u,u)<\infty\}$. $\mathbb{G}(\mathcal{E})$ satisfies the formula (2.23) for this choice of $(\mathcal{F}_e,\mathcal{E})$.

Let $\overset{\circ}{\mathcal{M}}_0(\mathbb{R})$ be the space of compactly supported finite signed measures on $\mathbb{R}$ and let $\overset{\circ}{\mathcal{M}}_{00}(\mathbb{R})=\{\mu\in\overset{\circ}{\mathcal{M}}_0(\mathbb{R}):\mu(\mathbb{R})=0\}$. The linear potential $U\mu$ of $\mu\in\overset{\circ}{\mathcal{M}}_0(\mathbb{R})$ is defined by $U\mu(\mathbf{x})=-\int_{\mathbb{R}}|x-y|\,\mu(dy),\ x\in\mathbb{R}$. By the Poisson equation in [F2, Th.4.3], $\mu\mapsto U\mu$ defines an injective map from $\overset{\circ}{\mathcal{M}}_{00}(\mathbb{R})$ into $H^1_e(\mathbb{R})$. For $\mu\in\overset{\circ}{\mathcal{M}}_{00}(\mathbb{R})$, we write $X_{U\mu}\in\mathbb{G}(\mathcal{E})$ as Z_μ. Then $\{Z_\mu:\mu\in\overset{\circ}{\mathcal{M}}_{00}(\mathbb{R})\}$ is a Gaussian field indexed by $\overset{\circ}{\mathcal{M}}_{00}(\mathbb{R})$ with covariance $\langle\mu,U\nu\rangle,\ \mu,\nu\in\overset{\circ}{\mathcal{M}}_{00}(\mathbb{R})$. This field is designated in [F2] as $\mathbb{G}(\mathbb{R})$, which can be identified with a sub-field of $\mathbb{G}(\mathcal{E})$.

The Cameron-Martin formula (2.24) holds for $\overset{\circ}{\mathcal{M}}_{00}(\mathbb{R}), H^1_e(\mathbb{R})$ and $\frac{1}{2}\mathbf{D}_{\mathbb{R}}$ in place of $\mathcal{M}_0,\mathcal{F}_e$ and $\mathcal{E}$, respectively. In particular, if we take $\nu_x=(\delta_x-\delta_0)/\sqrt{2}\in\overset{\circ}{\mathcal{M}}_{00}(\mathbb{R})$ for $x\in\mathbb{R}$ and write Z_{ν_x} as B_x, then $\mathbb{E}[B_xB_y]=\frac{1}{2}(|x|+|y|-|x-y|)$ so that $\{B_x:x\in\mathbb{R}\}$ is the standard Brownian motion with time parameter $x\in\mathbb{R}$ ([F2, §5.3]). For any $u\in H^1_e(\mathbb{R})$, the Gaussian random variable $X_u\in\mathbb{G}(\mathcal{E})$ can then be expressed as the Wiener integral $\int_{\mathbb{R}}u'(x)dB_x$ multiplied by $1/\sqrt{2}$. Hence, given any $x_1,x_2,\cdots,x_n\in\mathbb{R}$, any $u\in H^1_e(\mathbb{R})$ and any bounded Borel function H on $\mathbb{R}^n$, the identity (2.24) for $\nu_{x_i},1\le i\le n$, and $\sqrt{2}u$ in place of $\nu_i,1\le i\le n$, and u, repectively, reads

$$\begin{aligned}&\mathbb{E}\left[H(B_{x_1}+(u(x_1)-u(0)),\cdots,B_{x_n}+(u(x_n)-u(0)))\right]\\&=e^{-\frac{1}{2}\mathbf{D}_{\mathbb{R}}(u,u)}\ \mathbb{E}\left[\exp\left(\int_{\mathbb{R}}u'(x)dB_x\right)\ H(B_{x_1},\cdots,B_{x_n})\right],\end{aligned}\tag{2.26}$$

which is slightly more general than the original formula due to R.H.Cameron and W.T.Martin [CM].

3 Equilibrium potentials for recurrent Dirichlet forms

In this section, except for the last part below Lemma 3.8, we assume that $(\mathcal{E},\mathcal{F})$ is a regular recurrent Dirichlet form on $L^2(E;m)$ and satisfies the absolute continuity condition **(AC)**. We further make the following assumptions on the resolvent $\{R_\alpha,\ \alpha>0\}$ of the associated Hunt process $\mathbb{M}=(X_t,\mathbb{P}_x)$ on E:

(A.1) For any $B\in\mathcal{B}(E)$ with $m(B)>0$, $R_\alpha(x,B)>0$ for all $x\in E$,

(A.2) $R_\alpha f$ is lower semi-continuous for any non-negative Borel function f on E.

Condition **(A.1)** implies the irreducibility of the Dirichlet form $\mathcal{E}$.

For any bounded non-negative function w such that $\langle m,w\rangle>0$, consider the Dirichlet form $(\mathcal{E}^w,\mathcal{F})$ on $L^2(E;m)$ given by

$$\mathcal{E}^w(u,v)=\mathcal{E}(u,v)+(u,v)_{w\cdot m},\quad u,v\in\mathcal{F},\tag{3.1}$$

which is regular transient and associated with the canonical subprocess $\mathbb{M}^w$ of $\mathbb{M}$ with respect to the multiplicative functional $e^{-A_t},\ t\ge0$, for $A_t=\int_0^t w(X_s)ds,\ t\ge0$. Its extended Dirichlet space $\mathcal{F}^w_e$ coincides with $\mathcal{F}_e\cap L^2(E;w\cdot m)$ and its resolvent $R^w_\alpha f$ is expressed as

$$R^w_\alpha f(x)=\mathbb{E}_x\left[\int_0^\infty e^{-\alpha t-A_t}f(X_t)dt\right],\quad x\in E.$$

Let F be an admissible set, namely, a set satisfying (2.17), and B be any Borel set with positive m-measure. Their indicator functions 1_F and 1_B will be occasionally designated by g and h, respectively. The above notions for $w = g$ (resp. $w = h$) are denoted by $\mathcal{E}^g, \mathcal{F}_e^g, \mathbb{M}^g, R_\alpha^g$ (resp. $\mathcal{E}^h, \mathcal{F}_e^h, \mathbb{M}^h, R_\alpha^h$). R_0^g, R_0^h are denoted by R^g, R^h, respectively.

Lemma 3.1 (i) *$R^h g$ is bounded on E. For any bounded Borel function f on E vanishing outside a compact set, $R^g f$ is also bounded on E and moreover $R^g f$ is an element of $\mathcal{F}_e^g$ satisfying $\mathcal{E}^g(R^g f, v) = (f, v)$ for any $v \in \mathcal{F}_e^g$.*
(ii) *It holds that*

$$R^g(h \cdot R^h g)(x) = R^g g(x) - R^h g(x) + R^g(g \cdot R^h g)(x), \quad \text{for q.e. } x \in E. \tag{3.2}$$

The left hand side and the three terms of the right hand side are bounded functions in $\mathcal{F}_e^g$.

Proof. (i). **(A.1), (A.2)** imply that $\inf_{x\in K} R_2 h(x) = \ell(K) > 0$ for any compact set $K \subset E$, and consequently

$$R^h 1_K(x) \le \frac{1}{\ell(K)} R^h R_2 h(x) \le \frac{1}{\ell(K)} R^h R_1^h h(x) \le \frac{1}{\ell(K)} R^h h(x) = \frac{1}{\ell(K)}, \quad \forall x \in E. \tag{3.3}$$

The same bound holds for g in place of h and the boundedness assertions in (i) follow.

In view of [CF, Th.2.1.12] or [FO, Prop.2.5 (ii)], we see that, for a non-negative Borel function f on E, $R^w f \in \mathcal{F}_e^w$ if and only if $(f, R^w f) < \infty$ and in this case the Poisson equation $\mathcal{E}^w(R^w f, v) = (f, v),\ v \in \mathcal{F}_e^w$, is valid. In particular, the last assertion in (i) holds true.
(ii). First suppose $\overline{B}$ is compact. If we put $u = R^g(g + g \cdot R^h g - h \cdot R^h g)$, then $u \in \mathcal{F}_e^g$ by virtue of (i) and u satisfies the equation

$$\mathcal{E}^g(u, v) = (g + g \cdot R^h g - h \cdot R^h g, v), \quad v \in \mathcal{F} \cap C_c(E).$$

On the other hand, $R^h g \in \mathcal{F}_e^h \subset \mathcal{F}_e$ so that $R^h g \in \mathcal{F}_e \cap L^2(E; g \cdot m) = \mathcal{F}_e^g$ and

$$\mathcal{E}^g(R^h g, v) = \mathcal{E}^h(R^h g, v) + (R^h g, v)_{(g-h)m} = (g + gR^h g - hR^h g, v), \quad v \in \mathcal{F} \cap C_c(E).$$

Therfore $u = R^h g,\ m$-a.e., namely, (3.2) holds m-a.e.

For a general $B \in \mathcal{B}(E)$ with $m(B) > 0$, we put $B_n = B \cap U_n,\ h_n = 1_{B_n}$, for relatively compact open sets $\{U_n\}$ increasing to E. Then (3.2) holds m-a.e. for h_n in place of h and we have $(R^g v, h_n R^{h_n} g) = (v, R^g g - R^{h_n} g + R^g(g \cdot R^{h_n} g)), \quad v \in \mathcal{F} \cap C_c(E)$.
By noting the bound (3.3) and that $R^{h_n} g$ decreases to $R^h g$ as $n \to \infty$, we let $n \to \infty$ to get (3.2) holding m-a.e. together with the final statement of (ii). Since both hand sides are quasi-continuous by [FO, Prop.2.5 (ii)], we arrive at (3.2) holding q.e. □

Recall the Borel properly exceptional set $N \subset E$ of $\mathbb{M} = (X_t, \mathbb{P}_x)$ appearing in the absolute continuity condition **(AC)**. For $B \in \mathcal{B}(E \setminus N)$ of positive m-measure, we consider a *quasi-support* $\widetilde{B}$ of the measure $1_B \cdot m$, namely, the smallest quasi-closed set (up to the quasi-equivalence) outside which this measure vanishes. It is quasi-equivalent to the support of the corresponding PCAF $C_t = \int_0^t 1_B(X_s)ds,\ t \ge 0$, of $\mathbb{M}\big|_{E\setminus N}$ ([CF, Th.5.2.1]), so that we can and will make a specific choice of $\widetilde{B}$:

$$\widetilde{B} = \{x \in E \setminus N : \mathbb{P}_x(R = 0) = 1\}, \quad R(\omega) = \inf\{t > 0 : C_t(\omega) > 0\}. \tag{3.4}$$

$\widetilde{B}$ is a Borel subset of $E \setminus N$ ([FO, §3.2]) and hence $E \setminus N \setminus \widetilde{B}$ is finely open Borel set by enlarging the Borel properly exceptional set N of $\mathbb{M}$ if necessary. We denote by $R^{E\setminus\widetilde{B}}$ the 0-order resolvent kernel of the part process of $\mathbb{M}\big|_{E\setminus N}$ on this set.

In what follows, we fix an admissible set $F \subset E \setminus N$ and let $\{R\mu,\ \mu \in \mathcal{M}_0\}$ be the family of recurrent potentials relative to F.

Given a Borel set $B \subset E \setminus N$ with positive m-measure, $\widetilde{B}$ denotes the quasi-support of the measure $1_B \cdot m$.

Definition 3.2 A probability measure $\mu \in \mathcal{M}_0$ concentrated on $\widetilde{B}$ is called the *equilibrium measure for* $\widetilde{B}$ if $R\mu$ is constant q.e. on $\widetilde{B}$.

Define

$$\mu^{\widetilde{B}}(A) = \frac{1}{m(F)}\mathbb{P}_{1_F\cdot m}(X_{\sigma_{\widetilde{B}}} \in A), \quad A \in \mathcal{B}(E). \tag{3.5}$$

By virtue of Theorem 3.5.6 and (A.2.4) of [CF], we see that $\mu^{\widetilde{B}}$ is a probability measure on E concentrated on $\widetilde{B}$.

Lemma 3.3 *$\mu^{\widetilde{B}}$ belongs to the space $S_0^{g,(0)}$ of positive Radon measures of finite 0-order energy relative to the form $(\mathcal{E}^g, \mathcal{F})$.*

Proof. We first show that, for any non-negative Borel function u on E,

$$H_{\widetilde{B}}u(x) = H^g_{\widetilde{B}}u(x) + R^{E\setminus\widetilde{B}}(gH^g_{\widetilde{B}}u)(x), \tag{3.6}$$

where $H^g_{\widetilde{B}}$ denotes the counterpart of $H_{\widetilde{B}}$ for the process $\mathbb{M}^g$. In fact, by using the PCAF $G_t = \int_0^t g(X_s)ds$ of $\mathbb{M}$, we have for any non-negative bounded Borel function u,

$$\begin{aligned} R^{E\setminus\widetilde{B}}(1_F H^g_{\widetilde{B}}u)(x) &= \mathbb{E}\left[\int_0^{\sigma_{\widetilde{B}}} g(X_t)\mathbb{E}_{X_t}\left[e^{-G_{\sigma_{\widetilde{B}}}}u(X_{\sigma_{\widetilde{B}}})\right]dt\right] \\ &= \mathbb{E}_x\left[\int_0^{\sigma_{\widetilde{B}}} g(X_t)e^{-G_{\sigma_{\widetilde{B}}}\circ\theta_t}u(X_{\sigma_{\widetilde{B}}}\circ\theta_t)dt\right] = \mathbb{E}_x\left[\int_0^{\sigma_{\widetilde{B}}} g(X_t)e^{-(G_{\sigma_{\widetilde{B}}}-G_t)}u(X_{\sigma_{\widetilde{B}}})dt\right] \\ &= \mathbb{E}_x\left[e^{-G_{\sigma_{\widetilde{B}}}}u(X_{\sigma_{\widetilde{B}}})\left(e^{G_{\sigma_{\widetilde{B}}}}-1\right)\right] = H_{\widetilde{B}}u(x) - H^g_{\widetilde{B}}u(x). \end{aligned}$$

For a general non-negative Borel function, it suffices to aproximate by $u \wedge n$.

We next show that

$$R^{E\setminus\widetilde{B}}g(x) \text{ is bounded in } x \in E \setminus N. \tag{3.7}$$

Indeed, since $R = \sigma_{\widetilde{B}}$ a.s. ([CF, Proposition A.3.6]), $R^{E\setminus\widetilde{B}}g(x) = \mathbb{E}_x\left[\int_0^{\sigma_{\widetilde{B}}} g(X_t)dt\right]$ is dominated by $R^h g(x) = \mathbb{E}_x\left[\int_0^\infty e^{-C_t}g(X_t)dt\right]$ which is bounded on E by Lemma 3.1.

It follows from (3.5) and (3.6) that

$$\begin{aligned} \langle\mu^{\widetilde{B}}, R^g\mu^{\widetilde{B}}\rangle &= \frac{1}{m(F)}\left\langle g\cdot m, H^g_{\widetilde{B}}R^g\mu^{\widetilde{B}} + R^{E\setminus\widetilde{B}}(gH^g_{\widetilde{B}}R^g\mu^{\widetilde{B}})\right\rangle \\ &\le \frac{1}{m(F)}\left(\left\langle g\cdot m, R^g\mu^{\widetilde{B}}\right\rangle + \left\langle g\cdot m, R^{E\setminus\widetilde{B}}(gR^g\mu^{\widetilde{B}})\right\rangle\right). \end{aligned}$$

Observe that $\langle g\cdot m, R^g\mu^{\widetilde{B}}\rangle = \langle\mu^{\widetilde{B}}, R^g g\rangle = 1$ and

$$\left\langle g\cdot m, R^{E\setminus\widetilde{B}}(gR^g\mu^{\widetilde{B}})\right\rangle = \langle\mu^{\widetilde{B}}, R^g(g\cdot R^{E\setminus\widetilde{B}}g)\rangle \le \|R^{E\setminus\widetilde{B}}g\|_\infty.$$

which is finite by (3.7). Hence $\mu^{\widetilde{B}} \in S_0^{g,(0)}$ on account of [FO, Prop.2.5 (ii)]. □

Theorem 3.4 (i) *For any Borel set $B \subset E \setminus N$ of positive m-measure, the probability measure $\mu^{\widetilde{B}}$ defined by* (3.5) *is the unique equilibrium measure for the quasi-support $\widetilde{B}$ of $1_B \cdot m$. $R\mu^{\widetilde{B}}(x)$ takes a constant value $c(\widetilde{B})$ q.e. on $\widetilde{B}$ given by*

$$c(\widetilde{B}) = \frac{1}{m(F)^2}(1_F, R^{E\setminus\widetilde{B}}1_F)_m, \tag{3.8}$$

and $R\mu^{\widetilde{B}}$ admits the expression

$$R\mu^{\widetilde{B}} = c(\widetilde{B}) - \frac{1}{m(F)}R^{E\setminus\widetilde{B}}g, \quad \text{q.e. on } E. \tag{3.9}$$

(ii) $\mu^{\widetilde{B}}$ *is the unique measure among*

$$\{\mu \in \mathcal{S}_0^{g,(0)} : \mu(\widetilde{B}) = 1,\ \mu(E \setminus \widetilde{B}) = 0\} \tag{3.10}$$

minimizing $\mathcal{E}(R\mu, R\mu) = \langle \mu, R\mu \rangle$, and the minimum value equals $c(\widetilde{B})$.

Proof. (i). According to [FO, Th.3.5] about an explicit construction of the family $\{R\mu : \mu \in \mathcal{M}_0\}$ of recurrent potentials relative to the admissible set F, it holds that

$$Rf = H_F\check{R}(g \cdot R^g f) + R^g f - \frac{1}{m(F)}\langle m, f \rangle, \tag{3.11}$$

if a non-negative Borel function f on E satisfies $f \cdot m \in \mathcal{M}_0$, or equivalently, if f is m-integrable and $(f, R^g f) < \infty$. Here $\check{R}$ is an operator defined by [FO, (3.16)]. We make a special choice that $f = ph \cdot R^{ph}g$ for any constant $p \geq 1$ and $h = 1_B$, which satisfies just stated conditions in view of Lemma 3.1.

Notice that $\check{R}g = 0$, $R^g g = 1$, $R^{ph}(ph) = 1$, $g \cdot R^g(gR^{ph}g) = \check{R}_1(gR^{ph}g)$ by [FO, (3.13)], and $\check{R}_1\check{R}\varphi = \check{R}\varphi - (\check{R}_1\varphi - m(F)^{-1}\langle 1_F \cdot m, \varphi \rangle)$ by [FO, (3.16)]. (3.2) and (3.11) lead us to

$$\begin{aligned} R(ph \cdot R^{ph}g) &= H_F\check{R}R^g(ph \cdot R^{ph}g) + R^g(ph \cdot R^{ph}g) - \frac{1}{m(F)}\langle m, ph \cdot R^{ph}g \rangle \\ &= H_F\check{R}\left(g(1 - R^{ph}g + R^g(g \cdot R^{ph}g))\right) + 1 - R^{ph}g \\ &\qquad + R^g(g \cdot R^{ph}g) - \frac{1}{m(F)}(g, R^{ph}(ph)) \\ &= H_F\check{R}g - H_F\check{R}(g \cdot R^{ph}g) + H_F\check{R}\check{R}_1R^{ph}g - R^{ph}g + H_F\check{R}_1(g \cdot R^{ph}g) \\ &= -H_F\check{R}(g \cdot R^{ph}g) + H_F\check{R}(g \cdot R^{ph}g) - H_F\check{R}_1(g \cdot R^{ph}g) \\ &\qquad + m(F)^{-1}\langle 1_F \cdot m, R^{ph}g \rangle - R^{ph}g + H_F\check{R}_1(g \cdot R^{ph}g) \\ &= -R^{ph}g + m(F)^{-1}\langle m_F, R^{ph}g \rangle. \end{aligned} \tag{3.12}$$

We let $p \to \infty$. Since $R = \sigma_{\widetilde{B}}$ a.s. as was noted already, we have, for $x \in E \setminus N$,

$$\begin{aligned} &\lim_{p\to\infty} R^{ph}g(x) = \lim_{p\to\infty} \mathbb{E}_x\left[\int_0^\infty e^{-pC_t}g(X_t)dt\right] \\ &= \mathbb{E}_x\left[\int_0^{\sigma_{\widetilde{B}}} g(X_t)dt\right] + \lim_{p\to\infty} \mathbb{E}_x\left[\mathbb{E}_{X_{\sigma_{\widetilde{B}}}}[\int_0^\infty e^{-pC_t}g(X_t)dt]\right] = R^{E\setminus\widetilde{B}}g(x), \end{aligned}$$

$R^{ph}g(x)$ being bounded in x uniformly in $p \geq 1$ by Lemma 3.1.

Take any $v \in C_c(E)$. Then $v \cdot m \in \mathcal{M}_0$ and Rv is quasi-continuous and bounded by [FO, Th.3.5 (iv)] and Lemma 3.1. Let $\tau(t)$ be the right continuous inverse of the PCAF C_t: $\tau(t) = \inf\{s : C_s > t\}$. In particular, $\tau(0) = R = \sigma_{\widetilde{B}}$. By using [FO, Th.3.5 (ii)], we obtain

$$\begin{aligned}
&\lim_{p\to\infty}(v, R(ph \cdot R^{ph} g)) = \lim_{p\to\infty}(R^{ph}(ph \cdot Rv), g) \\
&= \lim_{p\to\infty} \mathbb{E}_{m_F}\left[\int_0^\infty e^{-pC_t} Rv(X_t) d(pC_t)\right] = \lim_{p\to\infty} \mathbb{E}_{m_F}\left[\int_0^\infty e^{-s} Rv(X_{\tau(s/p)}) ds\right] \\
&= \langle m_F, H_{\widetilde{B}} Rv\rangle = m(F)\langle \mu^{\widetilde{B}}, Rv\rangle = m(F)(R\mu^{\widetilde{B}}, v),
\end{aligned}$$

for the probability measure $\mu^{\widetilde{B}}$ defined by (3.5). Since $\mu^{\widetilde{B}} \in \mathcal{M}_0$ by Lemma 3.3, the last identity in the above is legitimated by (2.20).

Thus we have from (3.12) that

$$R\mu^{\widetilde{B}} = -\frac{1}{m(F)} R^{E\setminus\widetilde{B}} g + \frac{1}{m(F)^2}(1_F, R^{E\setminus\widetilde{B}} g), \quad \text{q.e. on } E. \tag{3.13}$$

Since $R^{E\setminus\widetilde{B}} g = 0$ on $\widetilde{B}$, $\mu^{\widetilde{B}}$ satisfies the condition of the equilibrium measure of $\widetilde{B}$ and, in fact, its potential $R\mu^{\widetilde{B}}$ takes the constant value (3.8) q.e. on $\widetilde{B}$ and (3.9) is valid.

To show the uniqueness of the equilibrium measure, assume that $\mu_1, \mu_2 \in \mathcal{M}_0$ are probability measures supported by $\widetilde{B}$ satisfying $R\mu_i = C_i$ q.e. on $\widetilde{B}$ for some constant C_i, for $i = 1, 2$. Since $R(\mu_1 - \mu_2) = C_1 - C_2$ q.e. on $\widetilde{B}$, $H_{\widetilde{B}} R(\mu_1 - \mu_2) = C_1 - C_2$ q.e. on E. Noting that $\mu_1 - \mu_2 \in \mathcal{M}_{00}$, for any $v \in \mathcal{F}_e \cap C_c(E)$, we have from (2.19),

$$0 = \mathcal{E}(H_{\widetilde{B}} R(\mu_1 - \mu_2), v) = \mathcal{E}(R(\mu_1 - \mu_2), H_{\widetilde{B}} v) = \langle \mu_1 - \mu_2, H_{\widetilde{B}} v\rangle = \langle \mu_1 - \mu_2, v\rangle.$$

Therefore, $\mu_1 = \mu_2$ which implies the uniqueness of the equilibrium measure.

(ii). Take any μ from the class (3.10) and put $\nu = \mu - \mu^{\widetilde{B}}$. Then $\langle \nu, R\nu\rangle = \langle \mu, R\mu\rangle - c(\widetilde{B})$ so that $\langle \mu, R\mu\rangle \geq c(\widetilde{B})$. The equality holds if and only if $\mathcal{E}(R\nu, R\nu) = \langle \nu, R\nu\rangle = 0$, or equivalently, $\mathcal{E}(R\nu, v) = 0$ for any $v \in \mathcal{F}_e$. As $\nu \in \mathcal{M}_{00}$, $\mathcal{E}(R\nu, v) = \langle \nu, \widetilde{v}\rangle$, $\forall v \in \mathcal{F}_e$, by (2.19), which completes the proof. □

We call $R\mu^{\widetilde{B}}$ *equilibrium potential* for $\widetilde{B}$ and $c(\widetilde{B})$ of (3.8) the *Robin constant* for $\widetilde{B}$ (relative to the admissible set F).

Remark 3.5 In establishing Theorem 3.4, we need to take, instead of a Borel set B itself, the quasi-support $\widetilde{B}$ of $1_B \cdot m$.
(i). If B is closed, then $\widetilde{B} \subset B$ and $m(B \setminus \widetilde{B}) = 0$.
(ii). If B is open, then $B \subset \widetilde{B} \subset \overline{B}$. In this case, $\widetilde{B} = \overline{B}$ if and only if every point of ∂B is regular for B.

Here we present a comparison statement of Robin constants for different recurrent Dirichlet forms. Let us consider two regular recurrent Dirichlet forms $(\mathcal{E}^{(i)}, \mathcal{F})$, $i = 1, 2$ on $L^2(E; m)$ both satisfying the condition **(AC)** with $N = \emptyset$ and conditions **(A.1)**, **(A.2)** as well. We assume that there exist some positive constants $\lambda \leq \Lambda$ with

$$\lambda \mathcal{E}^{(1)}(u, u) \leq \mathcal{E}^{(2)}(u, u) \leq \Lambda \mathcal{E}^{(1)}(u, u) \quad \text{for all } u \in \mathcal{F}$$

The two Dirichlet forms then share common notions of 'q.e.' and 'quasi-continuity'. For any Borel set $B \subset E$ with $m(B) > 0$, they have therefore a common quasi-support $\widetilde{B}$ of $1_B \cdot m$ up to q.e. equivalence. Let $F \subset E$ be a common admissible set for $\mathcal{E}^{(1)}$ and $\mathcal{E}^{(2)}$.

Proposition 3.6 *For any Borel set $B \subset E$, denote by $c^{(i)}(\widetilde{B})$ the Robin constant for $\widetilde{B}$ relative to F with respect to the Dirichlet form $\mathcal{E}^{(i)}$, $i = 1, 2$. Then*

$$\frac{1}{\Lambda}\, c^{(1)}(\widetilde{B}) \le c^{(2)}(\widetilde{B}) \le \frac{1}{\lambda}\, c^{(1)}(\widetilde{B}). \tag{3.14}$$

Proof. Let $\{R^{(i)}\mu, \mu \in \mathcal{M}_0^{(i)}\}$ be the family of recurent potentials relative to F with respect to $\mathcal{E}^{(i)}$, $i = 1.2$. Let $\mu_i (\in \mathcal{M}_0^{(i)})$ be the equilibrium measure of $\widetilde{B}$ with respect to $\mathcal{E}^{(i)}, i = 1, 2$. Since $R^{(1)}\mu_1 = c^{(1)}(\widetilde{B})$ q.e. on $\widetilde{B}$ which is also supported by the probability measure μ_2, we get by taking (2.19) into account, $c^{(1)}(\widetilde{B}) = \langle \mu_2, R^{(1)}\mu_1 \rangle = \mathcal{E}^{(2)}(R^{(2)}\mu_2, R^{(1)}\mu_1)$. The righthand side is dominated by

$$\begin{aligned} &\mathcal{E}^{(2)}(R^{(2)}\mu_2, R^{(2)}\mu_2)^{1/2}\, \mathcal{E}^{(2)}(R^{(1)}\mu_1, R^{(1)}\mu_1)^{1/2} \\ &\le c^{(2)}(\widetilde{B})^{1/2}\sqrt{\Lambda}\, \mathcal{E}^{(1)}(R^{(1)}\mu_1, R^{(1)}\mu_1)^{1/2} \le \sqrt{\Lambda} c^{(1)}(\widetilde{B})^{1/2} c^{(2)}(\widetilde{B})^{1/2}. \end{aligned}$$

Hence $c^{(1)}(\widetilde{B}) \le \Lambda c^{(2)}(\widetilde{B})$. The converse inequality follows similarly. □

For the sake of later use, let us state two formulae holding for recurrent potentials $R\mu$ relative to a fixed admissible set F. For F, define the probability measure $\widetilde{m}_F$ on E by

$$\widetilde{m}_F(A) = \frac{1}{m(F)} m(A \cap F), \quad A \in \mathcal{B}(E). \tag{3.15}$$

Besides the family $\{R\mu; \mu \in \mathcal{M}_0\}$ of recurent potentials relative to F, we consider a certain linear space $\widehat{\mathcal{M}}_0$ of finite signed measures on E such that each $\mu \in \widehat{\mathcal{M}}_{00} := \{\mu \in \widehat{\mathcal{M}}_0 : \mu(E) = 0\}$ admits a unique function $\widehat{R}\mu \in \mathcal{F}_e$ satisfying the Poisson equation $\mathcal{E}(\widehat{R}\mu, v) = \langle \mu, v \rangle$ for all $v \in \mathcal{F}_e \cap C_c(E)$. We assume that $\widetilde{m}_F \in \widehat{\mathcal{M}}_0$.

Lemma 3.7 *If μ is a probability measure and $\mu \in \mathcal{M}_0 \cap \widehat{\mathcal{M}}_0$, then, for* q.e. *$x \in E$,*

$$R\mu(x) = \widehat{R}(\mu - \widetilde{m}_F)(x) - \langle \widetilde{m}_F, \widehat{R}(\mu - \widetilde{m}_F) \rangle \tag{3.16}$$

Proof. Since $\mu - \widetilde{m}_F \in \mathcal{M}_0 \cap \widehat{\mathcal{M}}_0$ with zero total mass, we have by the Poisson equations $(R - \widehat{R})(\mu - \widetilde{m}_F) = c$ for some constant c. As $R\widetilde{m}_F = 0$ by (2.19), $R\mu(x) = \widehat{R}(\mu - \widetilde{m}_F)(x) + c$. Integrating both sides by $\widetilde{m}_F(dx)$, it holds that $\langle \widetilde{m}_F, \widehat{R}(\mu - \widetilde{m}_F) \rangle + c = 0$, yielding (3.16). □

Next let A be an open set with $A \cap F = \emptyset$, $(\mathcal{E}, \mathcal{F}_A)$ be the part of the Dirichlet form $(\mathcal{E}, \mathcal{F})$ on the set A and $\mathbb{M}_A = (X_t^A, \mathbb{P}_x)$ be the part of the Hunt process $\mathbb{M}$ on A. $\mathbb{M}_A$ is then a transient Hunt process on A associated with the regular Dirichlet form $(\mathcal{E}, \mathcal{F}_A)$ on $L^2(A; m)$. We denote by $\mathcal{S}_0^{A,(0)}$ the family of positive Radon measures on A of finite 0-order energy relative to $(\mathcal{E}, \mathcal{F}_A)$ and by $U^A\mu (\in \mathcal{F}_{e,A})$ the 0-order potential of $\mu \in \mathcal{S}_0^{A,(0)}$.
We further put $\mathcal{M}_0^{A,(0)} = \{\mu = \mu_1 - \mu_2 : \mu_1, \mu_2 \in \mathcal{S}_0^{A,(0)}\}$. We note that the inclusion $\mathcal{F}_e^A \subset \mathcal{F}_e^g$ holds because $(\mathcal{E}, \mathcal{F}^A)$ can be also considered as the part of $(\mathcal{E}, \mathcal{F}^g)$ on the set A.

The transition function P_t^A of $\mathbb{M}_A$ satisfies the absolute continuity condition **(AC)** holding for any $t > 0$ and $x \in A \setminus N$, N being the properly exceptional set of $\mathbb{M}$ appearing there. Hence the resolvent $\{R_\alpha^A, \alpha > 0\}$ of $\mathbb{M}_A$ admits the density function $r_\alpha^A(x, y)$, $x, y \in A \setminus N$, that is $e^{-\alpha t} P_t^A$-excessive in each variable. Define $r^A(x, y) = \lim_{\alpha \downarrow 0} r_\alpha^A(x, y)$ and put $R^A\mu(x) = \int_A r^A(x, y)\mu(dy)$. Then, exactly in the same way as the proof of [FO, Prop.2.5 (ii)], we can see that $\mu \in \mathcal{S}_0^{A,(0)}$ if and only if $\langle \mu, R^A\mu \rangle < \infty$ and in this case $R^A\mu$ is a quasi-continuous version of the potential $U^A\mu$.

Since $R^A f \le R^g f$ for any non-negative Borel function f, one can prove the inclusion

$$\{\mu \in \mathcal{S}_0^{g,(0)} : \mathrm{supp}[\mu] \subset A\} \subset \mathcal{S}_0^{A,(0)}, \tag{3.17}$$

in exactly the same way as the proof of [FO, Prop.2.5 (i)].

Lemma 3.8 *For any open set A with $A \cap F = \emptyset$ and for any measure $\mu \in \mathcal{M}_0$ concentrated on A, it holds that $\mu \in \mathcal{M}_0^{A,(0)}$ and*

$$H_{E\setminus A} R\mu = R\mu - R^A \mu. \tag{3.18}$$

Proof. In the proof of [FO; Th.3.8], we have seen that $\mu_{E\setminus A} \equiv \mu H_{E\setminus A} \in \mathcal{M}_0$ for any $\mu \in \mathcal{M}_0$. The first assertion is a consequence of (3.17). We show that

$$H_{E\setminus A} R\mu = R\mu_{E\setminus A}. \tag{3.19}$$

Let f be a bounded m-integrable function on E such that $R^g|f|$ is bounded. Then, by (2.17), $\langle \widetilde{m}_F, H_{E\setminus A} R\mu\rangle = \langle \widetilde{m}_F, R\mu\rangle = 0$ and further $\mathcal{E}(Rf, H_{E\setminus A} R\mu) = (f, H_{E\setminus A} R\mu)$, whose left hand side equals $\mathcal{E}(H_{E\setminus A} Rf, R\mu) = \langle \mu, H_{E\setminus A} Rf\rangle = \langle \mu_{E\setminus A}, Rf\rangle = (f, R\mu_{E\setminus A})$, arriving at (3.19).

Take any function $v \in \mathcal{F}_e^g$. In view of [CF, Th.3.4.8, Th.3.4.9], $H_{\mathbb{E}\setminus A}\widetilde{v} \in \mathcal{F}_e$ is $\mathcal{E}$-orthogonal to $\mathcal{F}_e^A$ and we get from (2.19) and (3.19)

$$\begin{aligned}&\mathcal{E}(H_{E\setminus A} R\mu + R^A\mu, v) = \mathcal{E}(R\mu_{E\setminus A} + R^A\mu, v)\\ &= \langle \mu_{E\setminus A}, v\rangle - \langle \mu_{E\setminus A}, 1\rangle\langle \widetilde{m}_F, v\rangle + \langle \mu, v - H_{E\setminus A} v\rangle = \langle \mu, v\rangle - \langle \mu.1\rangle\langle \widetilde{m}_F, v\rangle.\end{aligned}$$

Hence, the function $u = H_{E\setminus A} R\mu + R^A\mu$ is an element of $\mathcal{F}_e^g$ satisfying the first equation in (2.19). Since it also satisfies the normalization $\langle \widetilde{m}_F, u\rangle = \langle \widetilde{m}_F, R\mu\rangle = 0$, we obtain $u = R\mu$. □

In this section, we have considered the equilibrium measure and the equilibrium potential of a set for a recurrent Dirichlet form. For transient Dirichlet forms, these concepts have been introduced in a somewhat different way (see [De, Ch.4] and [FOT, §2.1, §2.2]). To be more precise, let $(\mathcal{E}, \mathcal{F})$ be a regular transient Dirichlet form on $L^2(E; m)$ and $\mathbb{M} = (X_t, \mathbb{P}_x)$ be the associated Hunt process on E. Let $\mathcal{S}_0^{(0)}$ be the family of positive Radon measures on E of finite 0-order energy and $U\mu \in \mathcal{F}_e$ be the 0-order potential of $\mu \in \mathcal{S}_0^{(0)}$ as were introduced in Section 2.2.

We consider any Borel set $B \subset E$ whose 0-order capacity $\mathrm{Cap}^{(0)}(B)$ is positive and finite. By the 0-order version of the second paragraph of [FOT, p 82] and [FOT, Th.4.3.3], there exists then a unique measure $\mu_B \in \mathcal{S}_0^{(0)}$ supported by $\overline{B}$ such that

$$\widetilde{U\mu_B}(x) = p_B(x) \text{ for q.e. } x \in E. \text{ where } p_B(x) = \mathbb{P}_x(\sigma_B < \infty),\ x \in E.$$

Note that $p_B = 1$ q.e. on B. μ_B (resp. $U\mu_B$) has been called the 0-order equilibrium measure (resp. equilibrium potential) of B.

Assume further that the set B is closed. Then $\mathrm{Cap}^{(0)}(B) = \langle \mu_B, p_B\rangle$ equals the total mass of μ_B and we can define the *renormalized equilibrium measure* μ^B of B by

$$\mu^B(A) = \mu_B(A)/\mathrm{Cap}^{(0)}(B), \quad A \in \mathcal{B}(E), \tag{3.20}$$

which is a probability measure concentrated on B. Accordingly, the *renormalized equilibrium potential* $\widetilde{U\mu^B}(x) = p_B(x)/\mathrm{Cap}^{(0)}(B)$, q.e. $x \in E$, takes a constant value $1/\mathrm{Cap}^{(0)}(B)$ q.e. on B so that this value can be regarded as the *Robin constant* for the closed set B relative to the renornmalized equilibrium measure μ^B.

We notice that, if we further assume that the Dirichlet form $\mathcal{E}$ is strongly local, then μ^B is concentrated on the boundary ∂B, because, for any function $\varphi \in \mathcal{F}_e \cap \mathbb{C}_c(E)$ with support in the interior of B, $\langle \mu^B, \varphi\rangle = \mathcal{E}(p_B, \varphi) = 0$.

4 GMCs via equilibrium potentials for recurrent forms

4.1 Properties of the family $\{\mu^{\mathbf{x},r}, f(\mathbf{x},r)\}$ for recurrent forms

Let E be either the whole plane $\mathbb{C}$ or the closure of the upper half-plane $\mathbb{H}$ and m be the Lebesgue measure on E. We consider a strongly local regular recurrent Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(E;m)$ and an associated diffusion process $\mathbb{M}=(X_t,\mathbb{P}_{\mathbf{x}})$ on E.
For $\mathbf{x}\in\mathbb{C}$, $s>0$, we let $B(\mathbf{x},s)=\{\mathbf{y}\in\mathbb{C}:|\mathbf{y}-\mathbf{x}|<s\}$ and $B(s)=B(\mathbf{0},s)$.
A function $u\in\mathcal{F}_e$ is said to be *$\mathcal{E}$-harmonic* on an open set $G\subset E$ if $\mathcal{E}(u,v)=0$ for any $v\in\mathcal{C}_G$ where $\mathcal{C}_G=\{v\in\mathcal{C}:\mathrm{supp}(v)\subset G\}$ for a special standard core $\mathcal{C}$ of $\mathcal{E}$.

We make the following assumption:

(B.1) The transition function P_t of $\mathbb{M}$ admits a density function $p_t(\mathbf{x},\mathbf{y})$ with respect to m satisfying the Gaussian estimate: there exist positive constants $K_i,\ k_i,\ i=1,2$, such that

$$\frac{K_1}{t}e^{-k_1|\mathbf{x}-\mathbf{y}|^2/t}\le p_t(\mathbf{x},\mathbf{y})\le\frac{K_2}{t}e^{-k_2|\mathbf{x}-\mathbf{y}|^2/t},\qquad \forall\mathbf{x},\mathbf{y}\in E, t>0. \tag{4.1}$$

Here are some important consequences of this assumption **(B.1)**. First, due to M.T.Barlow, A.Grigor'yan and T. Kumagai [BGK, Th.3.1, Cor.4.2], we have the following:

Proposition 4.1 (i) *$p_t(\mathbf{x},\mathbf{y})$ is positive and jointly continuous in $(t,\mathbf{x},\mathbf{y})\in(0,\infty)\times E\times E$.*
(ii) *For any $u\in\mathcal{F}_e$ that is $\mathcal{E}$-harmonic and bounded from below on an open set $G\subset E$, there exists its m-version $\widetilde{u}$ such that $\widetilde{u}$ is continuous on G. If $u\in\mathcal{F}_e$ is non-negative and $\mathcal{E}$-harmonic on $B(x,r)\subset E$, then $\widetilde{u}$ satisfies the Harnack inequality: there exists a constant C_H independent of $\mathbf{x}$ and r such that*

$$\sup\{\widetilde{u}(\mathbf{y}):\mathbf{y}\in B(\mathbf{x},r/2)\}\le C_H\inf\{\widetilde{u}(\mathbf{y}):\mathbf{y}\in B(\mathbf{x},r/2)\}. \tag{4.2}$$

Lemma 4.2 *For each $\mathbf{x}\in E$, the one-point set $\{\mathbf{x}\}$ is of zero capacity relative to $\mathcal{E}$.*

Proof. For $\alpha>0$, denote by $r_\alpha(\mathbf{x},\mathbf{y})$ (resp. $\widehat{r}_\alpha(\mathbf{x},\mathbf{y})$) the Laplace transform of $p_t(\mathbf{x},\mathbf{y})$ (resp. the transition density of the planar Brownian motion). By **(B.1)**, $2\pi K_1\widehat{r}_{2k_1}(\mathbf{x},\mathbf{y})\le r_1(\mathbf{x},\mathbf{y}),\ \mathbf{x},\mathbf{y}\in E$. Suppose $\mathrm{Cap}(\mathbf{y})=c>0$ for some $\mathbf{y}\in E$. According to [FOT], $\mathbb{E}_{\mathbf{x}}[e^{-\sigma_{\mathbf{y}}}]=c\cdot r_1(\mathbf{x},\mathbf{y})$ for q.e. $\mathbf{x}\in E$, which contradicts to the unboundedness of the righthand side due to the above inequality. □

For any Borel set $B\subset E$, a point $\mathbf{x}\in E$ is called *regular* for B if $\mathbb{P}_{\mathbf{x}}(\sigma_B=0)=1$. For an open set $G\subset E$, a point $\mathbf{x}\in\partial G$ is said to be *regular for the Dirichlet problem on G* if, for any bounded Borel function φ on ∂G that is continuous at $\mathbf{x}$, $\lim_{\mathbf{z}\to\mathbf{x},\mathbf{z}\in G}\mathbb{E}_{\mathbf{z}}[\varphi(X_{\sigma_{\partial G}});\sigma_{\partial G}<\infty]=\varphi(\mathbf{x})$.

Proposition 4.3 *Let G be an open subset of E. In the case that $E=\overline{\mathbb{H}}$, we assume that $\overline{G}\subset\mathbb{H}$. Then, a point $\mathbf{x}\in\partial G$ is regular for $E\setminus G$ if and only if $\mathbf{x}$ is regular for the Dirichlet problem on G.*

Proof. When G is bounded, the 'only if' part follows from E.B. Dynkin [Dy2, Th.13.1]. We reproduce a proof for a general open G under the current setting. For simplicity, we only consider the case that $E=\mathbb{C}$.

As the proof of [Dy1, Lem.6.3], we have for any $u>0$ and $\varepsilon>0$

$$\mathbb{P}_{\mathbf{x}}(|X_t-\mathbf{x}|\ge 2\varepsilon,\ \exists t\in[0,u]\cap\mathbb{Q})\le 2\sup_{t\le u,\mathbf{y}\in\mathbb{C}}P_t(\mathbf{y},\mathbb{C}\setminus B(\mathbf{y},\varepsilon)),\ \ \forall\mathbf{x}\in\mathbb{C}. \tag{4.3}$$

The righthand side tends to zero as $u \downarrow 0$ by the assumption **(B.1)**. Since $\mathbb{P}_{\mathbf{x}}(\sup_{t\in[0,u]} |X_t - \mathbf{x}| > \varepsilon)$ is dominated by the lefthand side of (4.3) with $\varepsilon/2$ in place of ε, we obtain for any $\varepsilon > 0$

$$\lim_{u\downarrow 0} \sup_{\mathbf{x}\in G} \mathbb{P}_{\mathbf{x}}(\sup_{t\le u} |X_t - \mathbf{x}| > \varepsilon) = 0. \tag{4.4}$$

On the other hand, assumption **(B.1)** along with Proposition 4.1 (i) implies that $\mathbb{M}$ is strong Feller in the sense that $P_t f \in C_\infty(\mathbb{C})$ for any bounded Borel function f on $\mathbb{C}$. According to Lemma 13.1 of [D2], this means that for $u > 0$ and $\tau_G = \sigma_{\mathbb{C}\setminus G}$

$$\mathbb{P}_{\mathbf{x}}(\tau_G > u) \quad \text{is upper semi continuous in} \quad \mathbf{x} \in \mathbb{C}. \tag{4.5}$$

Assume that $\mathbf{c} \in \partial G$ is regular for $\mathbb{C} \setminus G$, namely, $\mathbb{P}_{\mathbf{c}}(\tau_G = 0) = 1$. Take any bounded Borel function φ on ∂G which is continuous at $\mathbf{c}$ so that, for any $\varepsilon > 0$, there is $\alpha > 0$ with $|\varphi(\mathbf{y}) - \varphi(\mathbf{c})| < \varepsilon$ for any $\mathbf{y} \in B(\mathbf{c}, \alpha)$. We then get, for $f(\mathbf{x}) = \mathbb{E}_{\mathbf{x}}[\varphi(X_{\tau_G})]$, $\mathbf{x} \in G$,

$$|f(\mathbf{x}) - \varphi(\mathbf{c})| \le \varepsilon + 2||\varphi||_\infty (1 - \mathbb{P}_{\mathbf{x}}(X_{\tau_G} \in B(\mathbf{c}, \alpha))), \quad \mathbf{x} \in G. \tag{4.6}$$

By (4.4), we can find $u > 0$ with

$$\mathbb{P}_{\mathbf{x}}(\sup_{t\le u} |X_t - \mathbf{x}| \ge \alpha/2) < \varepsilon, \quad \forall \mathbf{x} \in G. \tag{4.7}$$

As $\mathbb{P}_{\mathbf{c}}(\tau_G > u) = 0$, (4.5) implies

$$\mathbb{P}_{\mathbf{x}}(\tau_G > u) < \varepsilon, \quad \forall \mathbf{x} \in B(\mathbf{c}, \delta), \quad \text{for some } \delta \in (0, \alpha/2). \tag{4.8}$$

It follows from (4.7) and (4.8) that, for any $\mathbf{x} \in G \cap B(\mathbf{c}, \delta)$,

$$\mathbb{P}_{\mathbf{x}}(\tau_G \le u, \ \sup_{t\le u} |X_t - \mathbf{x}| < \alpha/2) > 1 - 2\varepsilon.$$

Since the lefthand side is dominated by $\mathbb{P}_{\mathbf{x}}(|\mathbf{x} - X_{\tau_G}| < \alpha/2)$, we obtain $\mathbb{P}_{\mathbf{x}}(|\mathbf{c} - X_{\tau_G}| < \alpha) > 1 - 2\varepsilon$, which combined with (4.6) leads us to $|f(\mathbf{x}) - \varphi(\mathbf{c})| < \varepsilon + 4\varepsilon||\varphi||_\infty$, $\forall \mathbf{x} \in G \cap B(\mathbf{c}, \delta)$.

The 'if' part can be proved in exactly the same manner as [PS, Prop.3.6, Th.2.2] by noting that each one point set is polar by Lemma 4.2. □

We further make the next assumption:

(B.2) Let $B = B(\mathbf{x}, r) \cap E$ for any $\mathbf{x} \in E$, $r > 0$. In the case that $E = \overline{\mathbb{H}}$, we assume that $r \neq \Im \mathbf{x}$. Then every point of ∂B is regular for B and for $E \setminus \overline{B}$.

The property derived in Proposition 4.1(i) is much stronger than the absolute continuity condition **(AC)** which now holds with $N = \emptyset$. For any set B as in the assumption **(B.2)**, let $\widetilde{B}$ the quasi-support of $1_B \cdot m$ specified by (3.4). Then, by **(B.2)**

$$\widetilde{B} = \overline{B}, \quad \text{and} \quad \mathbb{P}_{\mathbf{x}}(\sigma_{\widetilde{B}} = \sigma_{\partial B}) = 1, \quad \text{for every } \mathbf{x} \in E \setminus \overline{B}. \tag{4.9}$$

Define $r_\alpha(\mathbf{x}, \mathbf{y}) = \int_0^\infty e^{-\alpha t} p_t(\mathbf{x}, \mathbf{y}) dt$, $\mathbf{x}, \mathbf{y} \in E$, and $R_\alpha f(\mathbf{x}) = \int_E r_\alpha(\mathbf{x}, \mathbf{y}) f(\mathbf{y}) m(d\mathbf{y})$, $\mathbf{x} \in E$. Then $\{R_\alpha, \alpha > 0\}$ is the resolvent of $\mathbb{M}$ satisfying the conditions **(A.1), (A.2)** in Section 3. Further, $r_1(\mathbf{x}, \mathbf{y})$ is positive and lower semi-continuous in $(\mathbf{x}, \mathbf{y}) \in E \times E$ so that $\inf_{\mathbf{x}\in F, \mathbf{y}\in F} r_1(\mathbf{x}, \mathbf{y}) > 0$ for any compact set $F \subset E$. Therefore any compact set $F \subset E$ with positive Lebesgue measure $m(F)$ can be an admissible set in the sense of (2.17). We make a special choice of it; for a fixed $S > 2$,

$$F = \overline{B(S+1)} \setminus B(S) \text{ when } E = \mathbb{C}; \ F = (\overline{B(S+1)} \setminus B(S)) \cap \overline{\mathbb{H}} \text{ when } E = \overline{\mathbb{H}}. \tag{4.10}$$

By the assumption **(B.2)**, we see that $\widetilde{F} = F$.

In what follows, we only deal with the case that $E = \mathbb{C}$ for simplicity of presentation and we aim at constructing the Gaussian multiplicative chaos on $B(S-1)$. But, in the case that $E = \overline{\mathbb{H}}$, all the statements below hold only by changing $B(S-1)$ into $B(S-1) \cap \{\mathbf{x} \in \mathbb{H} : \Im \mathbf{x} > 1\}$.

For the annulus F in $\mathbb{C}$ defined by (4.10), its indicator function 1_F will be denoted by g. Recall the related objects $\mathcal{E}^g$, $\mathcal{F}^g_e$, $\mathcal{S}_0^{g,(0)}$, $\mathbb{M}^g$, R^g, introduced in subsection 2.3 and section 3. We shall also use the notations $m_F(A) = m(F \cap A)$, $A \in \mathcal{B}(\mathbb{C})$, $\widetilde{m}_F = m_F/m(F)$. As the transition function P_t^g of $\mathbb{M}^g$ satisfies the absolute continuity condition **(AC)** with $N = \emptyset$, there exists a non-negative symmetric Borel measurable function $r^g(\mathbf{x}, \mathbf{y})$, $\mathbf{x}, \mathbf{y} \in \mathbb{C}$, such that it is $\mathbb{M}^g$-excessive in each variable and

$$R^g f(\mathbf{x}) = \int_{\mathbb{C}} r^g(\mathbf{x}, \mathbf{y}) f(\mathbf{y}) m(d\mathbf{y}) < \infty \quad \forall \mathbf{x} \in \mathbb{C}, \tag{4.11}$$

for any non-negative bounded Borel function f on $\mathbb{C}$ vanishing outside a bounded set by virtue of Lemma 3.1 and the definition [FO, (2.12)] of $r^g(\mathbf{x}, \mathbf{y})$. (4.11) particularly implies that, for each $\mathbf{x} \in \mathbb{C}$, $r^g(\mathbf{x}, \mathbf{y}) < \infty$ for q.e. $\mathbf{y} \in \mathbb{C}$ on account of [CF, Th.A.2.13 (v)]. Further, for any $\mu \in \mathcal{S}_0^{g.(0)}$, the function $R^g\mu$ define by $R^g\mu(\mathbf{x}) = \int_{\mathbb{C}} r^g(\mathbf{x}, \mathbf{y})\mu(d\mathbf{y})$, $\mathbf{x} \in \mathbb{C}$, is $\mathbb{M}^g$-excessive and a quasi continuous version of the 0-order potential $U^g\mu \in \mathcal{F}^g_e$ of μ in view of [FO, Prop.2.5].

Recall the space of signed measures $\mathcal{M}_0$ defined by (2.18). For any $\mu \in \mathcal{M}_0$, the recurrent potential $R\mu$ of μ relative to the admissible set F has been constructed in [FO, Th.3.5] explicitly by the formula

$$R\mu = H_F \check{R}(1_F R^g \mu) + R^g\mu - \frac{\mu(\mathbb{C})}{m(F)}. \tag{4.12}$$

which is a specific quasi continuous function in $\mathcal{F}^g_e$ satisfying the condition (2.19). Here H_F is defined by $H_F u(\mathbf{x}) = \mathbb{E}_x[u(X_{\sigma_F})]$, $\mathbf{x} \in \mathbb{C}$, and $\check{R}$ is a bounded operator on $L^2(F; m_F)$ to be explained below.

Let $(\check{R}_p)_{p>0}$ be the resolvent of the time changed process $\check{\mathbb{M}} = (X_{\tau_t}, \{\mathbb{P}_\mathbf{x}\}_{\mathbf{x} \in F})$ on F of $\mathbb{M}$ by its positive continuous additive functional $C_t = \int_0^t 1_F(X_s)ds$. τ_t is the right continuous inverse of C_t. F coincides with the support of C_t. We note that $\check{R}_1(\mathbf{x}, \cdot)$ is absolutely continuous with resepct to m_F and satisfies

$$\check{R}_1\varphi(\mathbf{x}) = \int_F \check{r}_1(\mathbf{x}, \mathbf{y})\varphi(\mathbf{y}) m_F(d\mathbf{y}), \quad \check{r}_1(\mathbf{x}, \mathbf{y}) = r^g(\mathbf{x}, \mathbf{y}) 1_F(\mathbf{y}), \quad \mathbf{x} \in F. \tag{4.13}$$

Define $\check{R}_1^1 = \check{R}_1$ and $\check{R}_1^n\varphi(\mathbf{x}) = \int_F \check{R}_1(\mathbf{x}, d\mathbf{y})\check{R}_1^{n-1}\varphi(\mathbf{y})$, $n \geq 2$. Then

$$\check{R}\varphi = \sum_{n=1}^{\infty} (\check{R}_1^n \varphi - \langle \widetilde{m}_F, \varphi \rangle), \quad \varphi \in L^2(F; m_F), \tag{4.14}$$

is convergent in $L^2(F; m_F)$. We also note that $\check{R}$ admits the bounds (cf. [FO, (3.7),(3.10)]):

$$||\check{R}\varphi||_2 \leq c_1 ||\varphi||_2, \quad ||\check{R}\varphi||_\infty \leq c_2 ||\varphi||_\infty, \quad \text{for some constants } c_1 > 0,\ c_2 > 0. \tag{4.15}$$

We have seen in Theorem 3.4 that, for any $B \in \mathcal{B}(\mathbb{C})$ with $m(B) > 0$, the quasi-support $\widetilde{B}$ of $1_B \cdot m$ admits the equilibrium measure $\mu^{\widetilde{B}} \in \mathcal{S}_0^{g,(0)}$ defined by (3.5). Let $B(\mathbf{x}, r)$ be any open disk with center $\mathbf{x} \in B(S-1)$ and radius $0 < r < 1$. Its closure will be denoted by $\bar{B}(\mathbf{x}, r)$. By (4.9), we have $\widetilde{B(\mathbf{x}, r)} = \overline{B}(\mathbf{x}, r)$. We denote the equilibrium measure $\mu^{\overline{B}(\mathbf{x}, r)}$ by $\mu^{\mathbf{x}, r}$. We then see from Theorem 3.4 that

$$\begin{cases} \mu^{\mathbf{x},r}(A) = \mathbb{P}_{\widetilde{m}_F}(X_{\sigma_{\partial B(\mathbf{x},r)}} \in A), \quad A \in \mathcal{B}(\partial B(\mathbf{x},r)), \\ R\mu^{\mathbf{x},r}(\mathbf{y}) = f(\mathbf{x},r) - \frac{1}{m(F)}\mathbb{E}_{\mathbf{y}}\left[\int_0^{\sigma_{\overline{B}(\mathbf{x},r)}} 1_F(X_t)dt\right], \quad \text{q.e. } \mathbf{y} \in \mathbb{C}, \quad \text{where} \\ f(\mathbf{x},r) := \frac{1}{m(F)}\mathbb{E}_{\widetilde{m}_F}\left[\int_0^{\sigma_{\partial B(\mathbf{x},r)}} 1_F(X_t)dt\right], \end{cases} \tag{4.16}$$

where $\widetilde{m}_F$ is the probability measure defined by (3.15). Thus $\mu^{\mathbf{x},r}$ is a probability measure concentrated on $\partial B(\mathbf{x},r)$ and its potential $R\mu^{\mathbf{x},r}$ takes a constant value $f(\mathbf{x},r)$ (called the Robin constant for $\bar{B}(\mathbf{x},r)$) q.e. on $\overline{B}(\mathbf{x},r)$.

Lemma 4.4 *Define* $\widehat{R}\mu^{\mathbf{x},r}(\mathbf{y}),\ \mathbf{y} \in \mathbb{C}$, *by*

$$\widehat{R}\mu^{\mathbf{x},r}(\mathbf{y}) = f(\mathbf{x},r) - \frac{1}{m(F)}v(\mathbf{y}), \text{ where } v(\mathbf{y}) = \mathbb{E}_{\mathbf{y}}\left[\int_0^{\sigma_{\overline{B}(\mathbf{x},r)}} 1_F(X_t)dt\right]. \tag{4.17}$$

$\widehat{R}\mu^{\mathbf{x},r}$ *is a quasi continuous version of* $R\mu^{\mathbf{x},r}$. *Further* $\widehat{R}\mu^{\mathbf{x},r}(\mathbf{y})$ *is continuous in* $\mathbf{y} \in B(S)$, $\mathcal{E}$-*harmonic on* $B(S) \setminus \overline{B}(\mathbf{x},r)$ *and identically equal to* $f(\mathbf{x},r)$ *on* $\overline{B}(\mathbf{x},r)$.

Proof. (4.16) implies the first statement. $v \in \mathcal{F}_e$ by (4.16) and v is bounded on $\mathbb{C}$ in view of (3.7). Take any disk B with $\overline{B} \subset B(S) \setminus \overline{B}(\mathbf{x},r)$. Then v is $\mathbb{M}$-harmonic on B in the sence that, for any open O with $\overline{O} \subset B$, $\mathbb{E}_{\mathbf{x}}[v(X_{\tau_O})] = v(\mathbf{x}),\ \mathbf{x} \in O$. Therefore, by virtue of [CF, Th.6.7.13], v is $\mathcal{E}$-harmonic on B, and by Proosition 4.1 (ii), there exists an m-version $\widetilde{v}$ of v which is continuous on B.

On the other hand, v is excessive relative to the part $\mathbb{M}_{\mathbb{C}\setminus\overline{B}(\mathbf{x},r)}$ of $\mathbb{M}$ on $\mathbb{C} \setminus \overline{B}(\mathbf{x},r)$ and so relative to the part $\mathbb{M}_B$ of $\mathbb{M}$ on B. Denote by P_t^B the transition function of $\mathbb{M}_B$. By taking the assumption **(B.1)** into account, we have $\widetilde{v}(\mathbf{y}) = \lim_{t\to 0} P_t^B\widetilde{v}(\mathbf{y}) = \lim_{t\to 0} P_t^B v(\mathbf{y}) = v(\mathbf{y})$, for all $\mathbf{y} \in B$, namely, $v(\mathbf{y})$ is continuous in $\mathbf{y} \in B$.

Denote v by v_r. v_r is identically zero on $\overline{B}(\mathbf{x},r)$ by assumption **(B.2)**. So it remains to prove that, for any $\mathbf{y}_0 \in \partial B(\mathbf{x},r)$,

$$\lim_{\mathbf{y}\to\mathbf{y}_0,\ \mathbf{y}\in B(S)\setminus\overline{B}(\mathbf{x},r)} v_r(\mathbf{y}) = 0. \tag{4.18}$$

Take $s \in (0,r)$. Then $v_s(z)$ is continuous in $z \in B(S) \setminus \overline{B}(\mathbf{x},s)$ and

$$v_s(\mathbf{y}) = v_r(\mathbf{y}) + \mathbb{E}_{\mathbf{y}}[v_s(X_{\sigma_{\partial B(\mathbf{x},r)}})], \quad \mathbf{y} \in \mathbb{C} \setminus \overline{B}(\mathbf{x},r),$$

Hence (4.18) follows from assumption **(B.2)** and Proposition 4.3. □

We first study the relation between the next two properties **(P.1), (P.2)** of the equilibrium potentials $\{R\mu^{\mathbf{x},r}\}$:

(P.1) There exist constants $\kappa \geq 1$ and $C_1 \geq 0$ such that for all $\mathbf{x} \in B(S-1)$ and $0 < 4r \leq t \leq \frac{1}{3}$,

$$\max\{\widehat{R}\mu^{\mathbf{x},r}(\mathbf{y}) : \mathbf{y} \in \partial B(\mathbf{x},t)\} \leq \kappa \min\{\widehat{R}\mu^{\mathbf{x},r}(\mathbf{y}) : \mathbf{y} \in \partial B(\mathbf{x},t)\} + C_1. \tag{4.19}$$

(P.2) There exist constants $\kappa \geq 1$ and $C_1 \geq 0$ such that, for all $\mathbf{x},\mathbf{y} \in B(S-1)$ and $0 < \varepsilon \leq \delta$ with $6\delta \leq |\mathbf{x}-\mathbf{y}| < 1/3$,

$$\langle \mu^{\mathbf{x},\varepsilon}, R\mu^{\mathbf{y},\delta} \rangle \leq \kappa f(\mathbf{y}, |\mathbf{x}-\mathbf{y}| - (\varepsilon+\delta)) + C_1. \tag{4.20}$$

Lemma 4.5 (i) *For $r \in (\varepsilon, 1/2)$,*

$$\max\{\widehat{R}\mu^{\mathbf{x},\varepsilon}(\mathbf{y}) : \mathbf{y} \in B(S) \setminus B(\mathbf{x}, r)\} = \max\{\widehat{R}\mu^{\mathbf{x},\varepsilon}(\mathbf{y}) : \mathbf{y} \in \partial B(\mathbf{x}, r)\}. \tag{4.21}$$

(ii) *If property* **(P.1)** *holds for some constant $\kappa > 0$ and $C_1 \geq 0$, then so does property* **(P.2)** *for the same constants κ, C_1.*

Proof. (i) Let $v(\mathbf{y})$, $\mathbf{y} \in B(S) \setminus \overline{B}(\mathbf{x}, \varepsilon)$, be the function defined by (4.17) for ε in place of r. Then, for $r \in (\varepsilon, 1/2)$ and for any $\mathbf{y} \in B(S) \setminus B(\mathbf{x}, r)$, $\mathbb{P}_{\mathbf{y}}(\sigma_{\partial B(\mathbf{x},r)} < \sigma_{\partial B(\mathbf{x},\varepsilon)}) = 1$ and so

$$v(\mathbf{y}) \geq \mathbb{E}_{\mathbf{y}}\left[\int_{\sigma_{\partial B(\mathbf{x},r)}}^{\sigma_{\partial B(\mathbf{x},\varepsilon)}} 1_F(X_t)dt\right] = \mathbb{E}_{\mathbf{y}}\left[v(X_{\sigma_{\partial B(\mathbf{x},r)}})\right] \geq \min_{\mathbf{z} \in \partial B(\mathbf{x},r)} v(\mathbf{z}),$$

yielding (4.21).
(ii) For any $\mathbf{z} \in B(\mathbf{x}, \varepsilon)$, we have $1/2 > |\mathbf{x} - \mathbf{y}| + \varepsilon \geq |\mathbf{z} - \mathbf{y}| \geq |\mathbf{x} - \mathbf{y}| - \varepsilon - \delta \geq 4\delta$ so that (i), (4.19) with $r = \delta$, $t = |\mathbf{x} - \mathbf{y}| - (\varepsilon + \delta) < 1/3$ and (2.20) imply

$$\begin{aligned}
\widehat{R}\mu^{\mathbf{y},\delta}(\mathbf{z}) &\leq \max\{\widehat{R}\mu^{\mathbf{y},\delta}(\mathbf{z}) : \mathbf{z} \in \partial B(\mathbf{y}, |\mathbf{x} - \mathbf{y}| - (\varepsilon + \delta))\} \\
&\leq \kappa \min\{\widehat{R}\mu^{\mathbf{y},\delta}(\mathbf{z}) : \mathbf{z} \in \partial B(\mathbf{y}, |\mathbf{x} - \mathbf{y}| - (\varepsilon + \delta))\} + C_1 \\
&\leq \kappa \int \widehat{R}\mu^{\mathbf{y},\delta}(\mathbf{w})\mu^{\mathbf{y},|\mathbf{x}-\mathbf{y}|-(\varepsilon+\delta)}(d\mathbf{w}) + C_1 = \kappa f(\mathbf{y}, |\mathbf{x} - \mathbf{y}| - (\varepsilon + \delta)) + C_1,
\end{aligned}$$

yielding (4.20). □

Lemma 4.6 (i) *For any $0 < r_0 < 1/3$, there exists a constant M_1 depending only on r_0 such that*

$$R^g\mu^{\mathbf{y},r}(\mathbf{x}) \leq M_1, \quad \text{for all } \mathbf{y} \in B(S-1),\ \mathbf{x} \in \mathbb{C} \setminus \overline{B}(\mathbf{y}, r_0),\ 0 < r < r_0/2. \tag{4.22}$$

(ii) *For any $0 < r_0 < 1/3$, there exists a constant M_2 depending only on r_0 such that*

$$|\widehat{R}\mu^{\mathbf{y},r}(\mathbf{x})| \leq M_2, \quad \text{for all } \mathbf{y} \in B(S-1),\ \mathbf{x} \in B(S) \setminus \overline{B}(\mathbf{y}, r_0),\ 0 < r < r_0/2. \tag{4.23}$$

(iii) *Property* **(P.1)** *holds true for some constant $\kappa > 0$ and $C_1 \geq 0$.*

Proof. (i) Let $G = B(S - 1/2) \setminus \overline{B}(\mathbf{y}, r_0/2)$ for any $\mathbf{y} \in B(S-1)$. Take any $r \in (0, r_0/2)$. Then, for any $v \in \mathcal{C}_G$, $\mathcal{E}(R^g\mu^{\mathbf{y},r}, v) = \mathcal{E}^g(R^g\mu^{y,r}.v) = \langle \mu^{y,r}, v\rangle = 0$, namely, $R^g\mu^{\mathbf{y},r}$ is $\mathcal{E}$-harmonic on G. As $R^g\mu^{\mathbf{y},r}$ is $\mathbb{M}^g$-excessive on $\mathbb{C}$, we can use Proposition 4.1 (ii) in the same way as the proof of Lemma 4.4 to conclude that it is continuous on G. We can then apply the Harnack inequality (4.2) to $R^g\mu^{\mathbf{y},r}$ to obtain for any $z \in G$, $s > 0$, with $B(\mathbf{z}, s) \subset G$ that

$$\max\{R^g\mu^{\mathbf{y},r}(\mathbf{w}) : \mathbf{w} \in B(\mathbf{z}, s/2)\} \leq C_H \min\{R^g\mu^{\mathbf{y},r}(\mathbf{w}) : \mathbf{w} \in B(\mathbf{z}, s/2)\}.$$

In particular, for any $\mathbf{z} \in \partial B(\mathbf{y}, r_0)$, $B(\mathbf{z}, r_0/2) \subset G$ and consequently,

$$\max\{R^g\mu^{\mathbf{y},r}(\mathbf{w}) : \mathbf{w} \in B(\mathbf{z}, r_0/4)\} \leq C_H \min\{R^g\mu^{\mathbf{y},r}(\mathbf{w}) : \mathbf{w} \in B(\mathbf{z}, r_0/4)\}.$$

Since the length of the circle $\partial B(\mathbf{y}, r_0)$ is $2\pi r_0$, $\partial B(\mathbf{y}, r_0)$ can be covered by $k = [2\pi r_0/(r_0/2)] + 1 = 13$ disks $B(\mathbf{z}_i, r_0/4)$ with center $\mathbf{z}_i \in \partial B(\mathbf{y}, r_0)$, $i = 1, 2, \cdots, 13$ such that $B(\mathbf{z}_i, r_0/4) \cap B(\mathbf{z}_{i+1}, r_0/4) \neq \emptyset$ for any $1 \leq i \leq 13$ with the convention that $\mathbf{z}_{14} = \mathbf{z}_1$. If we denote the maximum and the minimum of $R^g\mu^{\mathbf{y},r}$ on $\overline{B}(\mathbf{z}_i, r_0/4)$ by M_i and m_i, respectively, then $M_i/m_i \le C_H$ for any $1 \le i \le 13$.

Without loss of generality, we may assume that $M_1 = \max_{1\le i\le 13} M_i$. As $B(z_1, r_0/4)\cap B(z_2, r_0/4)$ is non-empty, we have $m_1 \le M_2$ and $M_1 \le C_H m_1 \le C_H M_2 \le C_H^2 m_2$ so that $M_1 \le C_H^2(m_1 \wedge m_2)$. Similarly, one can get, for any $1 \le i \le 7$, $M_1 \le C_H^i(m_1 \wedge m_2 \wedge \cdots \wedge m_i)$ and $M_1 \le C_H^i(m_{13} \wedge m_{12} \wedge \cdots \wedge m_{14-i})$. If the number k such that $m_1 \wedge m_2 \wedge \cdots \wedge m_{13} = m_k$ satisfies $1 \le k \le 7$ (resp. $7 \le k \le 13$), then the first (resp. second) relation with $i = 7$ yields that $M_1 \le C_H^7 m_k$. Let $\overline{D}(\mathbf{y}, r_0) = \cup_{i=1}^{13}\overline{B}(\mathbf{z}_i, r_0/4)$. We thus obtain, for any $r < r_0/2$,

$$\max\{R^g\mu^{\mathbf{y},r}(\mathbf{w}) : \mathbf{w} \in \overline{D}(\mathbf{y}, r_0)\} \le C_H^7 \min\{R^g\mu^{\mathbf{y},r}(\mathbf{w}) : \mathbf{w} \in \overline{D}(\mathbf{y}, r_0)\}. \tag{4.24}$$

Because of the inclusion $\partial B(\mathbf{y}, r_0) \subset \overline{D}(\mathbf{y}, r_0)$, we also have for any $r < r_0/2$

$$\max\{R^g\mu^{\mathbf{y},r}(\mathbf{w}) : \mathbf{w} \in \partial B(\mathbf{y}, r_0)\} \le C_H^7 \min\{R^g\mu^{\mathbf{y},r}(\mathbf{w}) : \mathbf{w} \in \partial B(\mathbf{y}, r_0)\}. \tag{4.25}$$

Since $R^g\mu^{\mathbf{y},r} \in \mathcal{F}_e^g$ is $\mathcal{E}^g$-harmonic on $\mathbb{C} \setminus \overline{B}(\mathbf{y}, r_0)$, $R^g\mu^{\mathbf{y},r}(\mathbf{x}) = H^g_{\overline{B}(\mathbf{y},r_0)}R^g\mu^{\mathbf{y},r}(\mathbf{x})$ for q.e. $\mathbf{x} \in \mathbb{C} \setminus \overline{B}(\mathbf{y}, r_0)$ by [FOT, Th.4.3.2], which holds for every $\mathbf{x} \in \mathbb{C} \setminus \overline{B}(\mathbf{y}, r_0)$ because the both hand sides are $\mathbb{M}^g$-excessive and consequently excessive relative to the part of $\mathbb{M}^g$ on $\mathbb{C} \setminus \overline{B}(\mathbf{y}, r_0)$. Accordingly we get, for any $\mathbf{x} \in \mathbb{C} \setminus \overline{B}(\mathbf{y}, r_0)$,

$$\begin{aligned} R^g\mu^{\mathbf{y},r}(\mathbf{x}) &= H^g_{\overline{D}(\mathbf{y},r_0)}R^g\mu^{\mathbf{y},r}(\mathbf{x}) \le \max\{R^g\mu^{\mathbf{y},r}(\mathbf{z}) : \mathbf{z} \in \overline{D}(\mathbf{y}, r_0)\} \\ &\le C_H^7 \min\{R^g\mu^{\mathbf{y},r}(\mathbf{z}) : \mathbf{z} \in \overline{D}(\mathbf{y}, r_0)\} \le \frac{C_H^7}{m(\overline{D}(\mathbf{y}, r_0))}\int_{\overline{D}(\mathbf{y},r_0)} R^g\mu^{\mathbf{y},r}(\mathbf{z})m(d\mathbf{z}) \\ &= \frac{C_H^7}{m(\overline{D}(\mathbf{y}, r_0))}\langle \mu^{\mathbf{y},r}, R^g 1_{\overline{D}(\mathbf{y},r_0)}\rangle. \end{aligned}$$

As the proof of Lemma 3.1 (i), $R^g 1_{\overline{D}(\mathbf{y},r_0)}(\mathbf{z}) \le 1/\ell(\overline{D}(\mathbf{y}, r_0))$ on $\mathbb{C}$ for

$$\ell(\overline{D}(\mathbf{y}, r_0)) = \inf\{R_2 g(\mathbf{x}) : \mathbf{x} \in \overline{D}(\mathbf{y}, r_0)\} \ge \inf\{r_2(\mathbf{x}, \mathbf{z}) : \mathbf{x} \in \overline{D}(\mathbf{y}, r_0), \mathbf{z} \in F\}m(F).$$

By the Gaussian lower bound in (4.1), the last term in the above is larger than $\overline{M}_1 m(F)$ with $\overline{M}_1 = \int_0^\infty (K_1/t)e^{-2t-k_1(2S-1/2)^2/t}dt < \infty$. Further $m(\overline{D}(\mathbf{y}, r_0))$ takes a positive value m_{r_0} independent of $\mathbf{y} \in B(S-1)$ and (4.22) holds for $M_1 = C_H^7/(\overline{M}_1 m(F)m_{r_0})$.
(ii) (4.22) particularly implies that $R^g\mu^{\mathbf{y},r}(\mathbf{x}) \le M_1$ for any $\mathbf{x} \in F$, $\mathbf{y} \in B(S-1)$ and $r \le r_0/2$. Hence, in view of (4.12) and (4.15), we have

$$||R\mu^{\mathbf{y},r}||_{L^\infty(\mathbb{C}\setminus B(\mathbf{y},r_0))} \le c_2 M_1 + M_1 + \frac{1}{m(F)} := M_2. \quad \mathbf{y} \in B(S-1),\ r \in (0, r_0/2).$$

By Lemma 4.4, $\widehat{R}\mu^{\mathbf{y},r}$ is a version of $R\mu^{\mathbf{y},r}$ and continuous on $B(S)$ so that we obtain the desired bound (4.23).
(iii) Take any $r_0 \in (0, 1/6)$ and let $G_1 = B(S-1/2) \setminus \overline{B}(\mathbf{y}, r_0)$. For $r < r_0$, $\widehat{R}\mu^{\mathbf{y},r}$ is $\mathcal{E}$-harmonic and continuous on G_1 by Lemma 4.4. For $r < r_0/2$, $|\widehat{R}\mu^{\mathbf{y},r}| \le M_2$ on G_1 by (ii) so that $\dot{R}\mu^{\mathbf{y},r} = \widehat{R}\mu^{\mathbf{y},r} + M_2$ is non-negative continuous and $\mathcal{E}$-harmonic on G_1. Hence the same method as in (i) works to obtain (4.25) for $\dot{R}\mu^{\mathbf{y},r}$ and $2r_0$ in place of $R^g\mu^{\mathbf{x},r}$ and r_0, respectively. Accordingly,

$$\max\{\widehat{R}\mu^{\mathbf{y},r}(\mathbf{z}) : \mathbf{z} \in \partial B(\mathbf{y}, 2r_0)\} \le C_H^7 \min\{\widehat{R}\mu^{\mathbf{y},r}(\mathbf{z}) : \mathbf{z} \in \partial B(\mathbf{y}, 2r_0)\} + C_1, \quad \forall r < r_0/2,$$

for some constant $C_1 \ge 0$, yielding (4.19) with $\kappa = C_H^7$. □

Proposition 4.7 (i) *There exists a symmetric Borel measurable function* $\mathfrak{r}(\mathbf{x},\mathbf{y}),\ \mathbf{x},\mathbf{y}\in\mathbb{C}$, *such that, for each* $\mathbf{x}\in\mathbb{C}$, *it is a difference of* $\mathbb{M}^g$*-excessive functions of* $\mathbf{y}\in\mathbb{C}$ *finite q.e. and*

$$R\mu(\mathbf{x})=\int_{\mathbb{C}}\mathfrak{r}(\mathbf{x},\mathbf{y})\mu(d\mathbf{y}),\quad \mathbf{x}\in\mathbb{C},\quad \text{for any } \mu\in\mathcal{M}_0. \tag{4.26}$$

(ii) *For any fixed* $0<\eta<1/2$, $\langle\mu^{\mathbf{x},r_1},R\mu^{\mathbf{y},r_2}\rangle$ *is uniformly bounded in* $r_1,r_2\in(0,\eta/8)$ *and* $\mathbf{x},\mathbf{y}\in B(S-1)$ *with* $|\mathbf{x}-\mathbf{y}|>\eta$.

$$\lim_{r_1,r_2\downarrow 0}\langle\mu^{\mathbf{x},r_1},R\mu^{\mathbf{y},r_2}\rangle=\mathfrak{r}(\mathbf{x},\mathbf{y}), \tag{4.27}$$

for $m\times m$*-a.e.* $(\mathbf{x},\mathbf{y})\in B(S-1)\times B(S-1)\cap\{(\mathbf{x},\mathbf{y}):|\mathbf{x}-\mathbf{y}|>\eta\}$.

A proof of this proposition will be given in Appendix (subsection 7.1). In the proof of the second assertion (ii), we shall make a full use of the upper Gaussian bound in assumption **(B.1)** along with Lemma 4.6.

Before going into our task of constructing Gaussian multiplicative chaos, we need to make an additional assumption that

(B.3) There exist a constant $C_2>0$ such that

$$\sup\{f(\mathbf{y},r):\mathbf{y}\in B(S-1)\}\le C_2\inf\{f(\mathbf{z},r):\mathbf{z}\in B(S-1)\},\quad r\in(0,1). \tag{4.28}$$

4.2 Construction of Gaussian multiplicative chaos from $\{\mu^{\mathbf{x},r},f(\mathbf{x},r)\}$

Let $\{X_u;u\in\mathcal{F}_e\}$ be the centered Gaussian field defined on a probability space $(\Omega,\mathcal{B},\mathbb{P})$ with covariance $\mathbb{E}[X_uX_v]=\mathcal{E}(u,v)\ u,v\in\mathcal{F}_e$. Define

$$Y^{\mathbf{x},r}=X_{R\mu^{\mathbf{x},r}},\qquad \mathbf{x}\in B(S-1),\quad 0<r<1, \tag{4.29}$$

for the equilbrium potential $R\mu^{\mathbf{x},r}\in\mathcal{F}_e$ with respect to the closed disk $\overline{B}(\mathbf{x},r)$.

On account of the probabilistic expression (4.16) of the Robin constant $f(\mathbf{x},r)$, we readily see that $f(\mathbf{x},r)$ is a strictly decreasing continuous function of $r\in(0,1)$. We denote its inverse function by $f^{-1}(\mathbf{x},r)$. For any $r\in(0,1/2)$, let $\overline{r}=\overline{r}(\mathbf{x})=f^{-1}(\mathbf{x},[f(\mathbf{x},r)])$, where $[f(\mathbf{x},r)]$ is the integer part of $f(\mathbf{x},r)$.
Given $\alpha>0$ and $0<\varepsilon<\varepsilon_0<1/2$, we shall consider the set $G^{\alpha,\varepsilon_0}_{\mathbf{x},\varepsilon}(\omega)$ defined by

$$G^{\alpha,\varepsilon_0}_{\mathbf{x},\varepsilon}(\omega)=\{Y^{\mathbf{x},\overline{r}}\le\alpha f(\mathbf{x},\overline{r}),\ \forall r\in(\varepsilon,\varepsilon_0)\}. \tag{4.30}$$

For a fixed $\gamma>0$, put

$$\widetilde{Y}^{\mathbf{x},\varepsilon,\gamma}=\gamma Y^{\mathbf{x},\varepsilon}-\frac{\gamma^2}{2}V(Y^{\mathbf{x},\varepsilon}), \tag{4.31}$$

where $V(Y^{\mathbf{x},\varepsilon})=f(\mathbf{x},\varepsilon)$ is the variance of $Y^{\mathbf{x},\varepsilon}$.

The following estimate is well known for a centered Gaussian random variable ξ with variance $V(\xi)$ (see [MR; Lem.5.1.3]):

$$\mathbb{P}[|\xi|>a]\le\exp\left(-\frac{a^2}{2V(\xi)}\right)\qquad\forall a>0. \tag{4.32}$$

Proposition 4.8 *For any* $\alpha>\gamma$ *and* $\varepsilon_0>0$, *there exists* $p(\alpha,\gamma,\varepsilon_0)>0$ *independent of* ε *and* $\mathbf{x}$ *such that* $\lim_{\varepsilon_0\to 0}p(\alpha,\gamma,\varepsilon_0)=0$ *and*

$$\mathbb{E}\left[e^{\widetilde{Y}^{\mathbf{x},\varepsilon,\gamma}}:\Omega\setminus G^{\alpha,\varepsilon_0}_{\mathbf{x},\varepsilon}\right]\le\ p(\alpha,\gamma,\varepsilon_0),\quad\text{for all }\varepsilon\in(0,\varepsilon_0)\text{ and }\mathbf{x}\in B(S-1).$$

Proof. Put $M_{\mathbf{x},\varepsilon,\gamma,\alpha} = \mathbb{E}\left[e^{\widetilde{Y}^{\mathbf{x},\varepsilon,\gamma}} : \Omega \setminus G^{\alpha,\varepsilon_0}_{\mathbf{x},\varepsilon}\right]$. Then

$$M_{\mathbf{x},\varepsilon,\gamma,\alpha} = e^{-(\gamma^2/2)V(Y^{\mathbf{x},\varepsilon})}\mathbb{E}\left[e^{\gamma Y^{\mathbf{x},\varepsilon}} : \bigcup_{k\in([f(\mathbf{x},\varepsilon_0)],[f(\mathbf{x},\varepsilon])} \{Y^{\mathbf{x},f^{-1}(\mathbf{x},k)} > \alpha k\}\right].$$

Since, for any $\varepsilon < f^{-1}(\mathbf{x},k)$,

$$\mathrm{cov}(\gamma Y^{\mathbf{x},\varepsilon}, Y^{\mathbf{x},f^{-1}(\mathbf{x},k)}) = \gamma\langle \mu^{\mathbf{x},\varepsilon}, R\mu^{\mathbf{x},f^{-1}(\mathbf{x},k)}\rangle = \gamma f(\mathbf{x}, f^{-1}(\mathbf{x},k)) = \gamma k,$$

the Cameron-Martin formula (2.25) applies in getting

$$\begin{aligned} M_{\mathbf{x},\varepsilon,\gamma,\alpha} &= \mathbb{P}\left(\bigcup_{k\in([f(\mathbf{x},\varepsilon_0)],[f(\mathbf{x},\varepsilon)])} \{Y^{\mathbf{x},f^{-1}(\mathbf{x},k)} + \gamma k > \alpha k\}\right) \\ &\leq \sum_{k\in([f(\mathbf{x},\varepsilon_0)],[f(\mathbf{x},\varepsilon)])} \mathbb{P}\left(Y^{\mathbf{x},f^{-1}(\mathbf{x},k)} > (\alpha-\gamma)k\right). \end{aligned}$$

As $Y^{\mathbf{x},f^{-1}(\mathbf{x},k)}$ is a centered Gaussian random variable with variance k, we get from (4.32),

$$M_{\mathbf{x},\varepsilon,\gamma,\alpha} \leq \frac{1}{2}\sum_{k=[f(\mathbf{x},\varepsilon_0)]}^{\infty} \exp\left(-\frac{((\alpha-\gamma)k)^2}{2k}\right) \leq \frac{1}{2}\sum_{k=k_0(\varepsilon_0)}^{\infty} \exp\left(-\frac{(\alpha-\gamma)^2 k}{2}\right),$$

where $k_0(\varepsilon_0) = \inf\{[f(\mathbf{z},\varepsilon_0)] : \mathbf{z} \in B(S-1)\}$.

We let $p(\alpha,\gamma,\varepsilon_0) = \sum_{k=k_0(\varepsilon_0)}^{\infty} e^{-(\alpha-\gamma)^2k/2}$. By (4.16), Lemma 4.2 and the recurrence of $\mathcal{E}$, it holds that

$$\lim_{r\downarrow 0} f(\mathbf{x},r) = \frac{1}{m(F)}\mathbb{E}_{\widetilde{m}_F}\left[\int_0^\infty 1_F(X_t)dt\right] = \infty, \quad \forall \mathbf{x} \in B(S-1).$$

On the other hand, the assumption **(B.3)** yields that, for a fixed $\mathbf{c} \in B(S-1)$,

$$f(\mathbf{c},r) \leq C_1 \inf\{f(\mathbf{z},r) : \mathbf{z} \in B(S-1)\}, \quad \text{for every } r \in (0,1).$$

Therefore $\lim_{\varepsilon_0\to 0} k_0(\varepsilon_0) = \infty$ and $\lim_{\varepsilon_0\to 0} p(\alpha,\gamma,\varepsilon_0) = 0$. □

Before proceeding further, let us prepare the following proposition about the existence of a measurable version in two variables $(\mathbf{x},\omega)$ of the random variable $Y^{\mathbf{x},r}(\omega)$ defined by (4.29) for each $r > 0$.

Proposition 4.9 *For any $r \in (0,1/2)$ and finite positive measure σ on $B(S-1)$, there exists a measurable function $Y^r(\mathbf{x},\omega)$ on $B(S-1)\times\Omega$ such that, for σ-a.e. $\mathbf{x} \in B(S-1)$, $Y^r(\mathbf{x},\omega) = Y^{\mathbf{x},r}(\omega)$ $\mathbb{P}$-a.s.*

Proof. Let us fix $r \in (0,1/2)$ and a finite positive measure σ on $B(S-1)$. We shall first prove that the map from $\mathbf{x} \in B(S-1)$ to $R\mu^{\mathbf{x},r} \in \mathcal{F}_e$ is continuous. Since

$$\mathcal{E}(R\mu^{\mathbf{x},r} - R\mu^{\mathbf{y},r}, R\mu^{\mathbf{x},r} - R\mu^{\mathbf{y},r}) = f(\mathbf{x},r) + f(\mathbf{y},r) - 2\langle \mu^{\mathbf{y},r}, R\mu^{\mathbf{x},r}\rangle,$$

it is enough to show that $\lim_{n\to\infty} f(\mathbf{y}_n,r) = f(\mathbf{x},r)$ and $\lim_{n\to\infty}\langle \mu^{\mathbf{y}_n,r}, R\mu^{\mathbf{x},r}\rangle = f(\mathbf{x},r)$ for any sequence $\mathbf{y}_n \in B(S-1)$ converging to $\mathbf{x} \in B(S-1)$, In view of (4.16), these two relations are equivalent to $\lim_{n\to\infty}(1_F, R^{E\setminus\bar{B}(\mathbf{y}_n,r)}1_F)_m = (1_F, R^{E\setminus\bar{B}(\mathbf{x},r)}1_F)_m$ and $\lim_{n\to\infty}\langle\mu^{\mathbf{y}_n,r}, R^{E\setminus\bar{B}(\mathbf{x},r)}1_F\rangle = 0$, respectively. But by (4.16) again,

$$\begin{aligned}\langle\mu^{\mathbf{y}_n,r}, R^{E\setminus\bar{B}(\mathbf{x},r)}g\rangle &= \mathbb{E}_{\widetilde{m}_F H_{\bar{B}(\mathbf{y}_n,r)}}\left[\int_0^{\sigma_{\bar{B}(\mathbf{x},r)}} 1_F(X_t)dt\right] \\ &= \mathbb{E}_{\widetilde{m}_F}\left[\int_{\sigma_{\bar{B}(\mathbf{y}_n,r)}}^{\sigma_{\bar{B}(\mathbf{y}_n,r)}+\sigma_{\bar{B}(\mathbf{x},r)}\circ\theta(\sigma_{\bar{B}(\mathbf{y}_n,r)})} 1_F(X_t)dt\right].\end{aligned}$$

Therefore, if we can show that, for any sequence $\mathbf{y}_n \in (S-1)$ converging to $\mathbf{x} \in B(S-1)$,

$$\mathbb{P}_{\mathbf{z}}(\lim_{n\to\infty}\sigma_{\bar{B}(\mathbf{y}_n,r)} = \lim_{n\to\infty}(\sigma_{\bar{B}(\mathbf{y}_n,r)} + \sigma_{\bar{B}(\mathbf{x},r)}\circ\theta(\sigma_{\bar{B}(\mathbf{y}_n,r)}) = \sigma_{\bar{B}(\mathbf{x},r)}) = 1,\ \forall\mathbf{z}\in E, \tag{4.33}$$

the desired continuity follows.

For any $0 < \varepsilon < r$, if $|\mathbf{x}-\mathbf{y}_n| < \varepsilon$ then $\overline{B}(\mathbf{x}, r-\varepsilon) \subset \overline{B}(\mathbf{y}_n, r)\cap\overline{B}(\mathbf{x}, r) \subset \overline{B}(\mathbf{y}_n, r)\cup\overline{B}(\mathbf{x}, r) \subset B(\mathbf{x}, r+\varepsilon)$. Hence $\sigma_{B(\mathbf{x},r+\varepsilon)} \le \underline{\lim}_{n\to\infty}\sigma_{\bar{B}(\mathbf{y}_n,r)} \le \overline{\lim}_{n\to\infty}\sigma_{\bar{B}(\mathbf{y}_n,r)} \le \sigma_{\overline{B}(\mathbf{x},r-\varepsilon)}$. The same relation also holds for $\sigma_{\bar{B}(\mathbf{y}_n,r)} + \sigma_{\bar{B}(\mathbf{x},r)}\circ\theta(\sigma_{\bar{B}(\mathbf{y}_n,r)})$ instead of $\sigma_{\bar{B}(\mathbf{y}_n,r)}$. Therefore, for the proof of (4.33), it only remains to show that $\mathbb{P}_{\mathbf{z}}(\lim_{k\to\infty}\sigma_{B(\mathbf{x},r+\varepsilon_k)} = \lim_{k\to\infty}\sigma_{\overline{B}(\mathbf{x},r-\varepsilon_k)} = \sigma_{\overline{B}(\mathbf{x},r)}) = 1$, $\forall\mathbf{z}\in E$ for some sequence $\varepsilon_k \downarrow 0$.

By **(B.2)**, $\sigma_{\overline{B}(\mathbf{x},r)} = \sigma_{B(\mathbf{x},r)}$ a.s. Clearly $\sigma_{\overline{B}(\mathbf{x},r-\varepsilon_k)} \ge \sigma_{B(\mathbf{x},r)}$. Further, if $\sigma_{B(\mathbf{x},r)}(\omega) < t$, then $X_s(\omega) \in B(\mathbf{x}, r)$ for some $s < t$ and hence $X_s(\omega) \in \overline{B}(\mathbf{x}, r-\varepsilon_k)$ for some $k \ge 1$, that is $\sigma_{\overline{B}(\mathbf{x},r-\varepsilon_k)} < t$. Therefore $\lim_{k\to\infty}\sigma_{\overline{B}(\mathbf{x},r-\varepsilon_k)} \le \sigma_{B(\mathbf{x},r)}$ and hence $\mathbb{P}_{\mathbf{z}}(\lim_{k\to\infty}\sigma_{\overline{B}(\mathbf{x},r-\varepsilon_k)} = \sigma_{B(\mathbf{x},r)} = \sigma_{\bar{B}(\mathbf{x},r)}) = 1$. To show another relation, put $\underline{\sigma} = \lim_{k\to\infty}\sigma_{B(\mathbf{x},r+\varepsilon_k)} \le \sigma_{\overline{B}(\mathbf{x},r)}$. Since $X_{\underline{\sigma}} = \lim_{k\to\infty}X_{\sigma_{B(\mathbf{x},r+\varepsilon_k)}} \in \cap_k\overline{B}(\mathbf{x}, r+\varepsilon_k) = \overline{B}(\mathbf{x}, r)$, $\underline{\sigma} \ge \sigma_{\overline{B}(\mathbf{x},r)}$. Hence we also have $\mathbb{P}_{\mathbf{z}}(\lim_{k\to\infty}\sigma_{B(\mathbf{x},r+\varepsilon_k)} = \sigma_{\overline{B}(\mathbf{x},r)}) = 1$. Thus we have the desired continuity.

As $\mathbb{E}[(Y^{\mathbf{x},r} - Y^{\mathbf{y},r})^2] = \mathcal{E}(R\mu^{\mathbf{x},r} - R\mu^{\mathbf{y},r}, R\mu^{\mathbf{x},r} - R\mu^{\mathbf{y},r})$, the continuity verified in the above implies that $\mathbf{x} \mapsto Y^{\mathbf{x},r}$ is a continuous map from $B(S-1)$ to $L^2(\mathbb{P})$, and consequently, a uniformly continuous map from K to $L^2(\mathbb{P})$ for any compact sunset K of $B(S-1)$. For $n \ge 1$, express $B(S-1)$ as a finite disjoint sum $B(S-1) = \sum_k C_{n,k}$, where $C_{n,k}$ is an intersection of $B(S-1)$ and a square of side length $1/n$. Pick the unique point $\mathbf{c}_{n,k}$ from $\overline{C_{n,k}}$ with shortest distance from the origin and let $Y_n^r(\mathbf{x},\omega) = \sum_k 1_{C_{n,k}}(\mathbf{x})Y^{\mathbf{c}_{n,k},r}(\omega)$. Then $Y_n^r(\mathbf{x},\omega)$ is measurable in $(\mathbf{x},\omega) \in B(S-1)\times\Omega$ and, by the stated uniform continuity, we have for any compact subset K of $B(S-1)$

$$\lim_{n\to\infty}\sup_{\mathbf{x}\in K}\mathbb{E}[(Y_n^r(\mathbf{x},\omega) - Y^{\mathbf{x},r}(\omega))^2] = 0. \tag{4.34}$$

Consequently,

$$\begin{aligned}&\lim_{n,\ell\to\infty}\int_{\Omega\times K}(Y_n^r(\mathbf{x},\omega) - Y_\ell^r(\mathbf{x},\omega))^2\mathbb{P}(d\omega)\sigma(d\mathbf{x}) \\ &\le \sigma(K)\lim_{n,\ell\to\infty}\sup_{\mathbf{x}\in K}\mathbb{E}[(Y_n^r(\mathbf{x},\omega) - Y_\ell^r(\mathbf{x},\omega))^2] = 0,\end{aligned}$$

namely, $\{Y_n^r(\mathbf{x},\omega)\}$ is a Cauchy sequence in $L^2(\Omega\times K, \mathbb{P}\times\sigma)$.

By choosing a suitable subsequence $\{n'\}$ of $\{n\}$, $Y_{n'}(\mathbf{x},\omega)$ conveges to a $\mathbb{P}\times\sigma$-measurable function $Y^r(\mathbf{x},\omega)$ on $K\times\Omega$ as $n' \to \infty$. In particular, for σ-a.e. $\mathbf{x} \in K$, $\lim_{n'\to\infty}Y_{n'}^r(\mathbf{x},\omega) = Y^r(\mathbf{x},\omega)$, $\mathbb{P}$-a.s., which combined with (4.34) yields $\mathbb{E}[(Y^r(\mathbf{x},\omega) - Y^{\mathbf{x},r}(\omega))^2] = 0$ and $Y^r(\mathbf{x},\omega) = Y^{\mathbf{x},r}(\omega)$ $\mathbb{P}$-a.s., for σ-a.e. $\mathbf{x} \in K$. Since K is an arbitrary compact subset of $B(S-1)$, the proof of Proposition 4.9 is complete. □

Throughout the rest of this subsection, we shall consider a positive measure σ on $B(S-1)$ absolutely continuous with respect to the Lebesgue measure with an integrable density on $B(S-1)$. For any $A \in \mathcal{B}(B(S-1))$ and for $0 < \varepsilon < \varepsilon_0$, define

$$I_\varepsilon(\omega) = \int_A e^{\widetilde{Y}^{\mathbf{x},\varepsilon,\gamma}} \sigma(dx), \quad J_\varepsilon(\omega) = \int_A e^{\widetilde{Y}^{\mathbf{x},\varepsilon,\gamma}} 1_{G^{\alpha,\varepsilon_0}_{\mathbf{x},\varepsilon}} \sigma(dx), \tag{4.35}$$

where the random variable $Y^{\mathbf{x},\varepsilon}(\omega)$ involved in $\widetilde{Y}^{\mathbf{x},\varepsilon,\gamma}$ and $G^{\alpha,\varepsilon_0}_{\mathbf{x},\varepsilon}$ is taken to be its measurable version in Proposition 4.9 so that the integrals in (4.35) make a perfect sense and they can be regarded as a random measures on $(B(S-1), \mathcal{B}(B(S-1)))$.

We aim at deriving the convergence in probability of the random measure I_ε as $\varepsilon \downarrow 0$ toward a non-degenerate random measure on $B(S-1)$ in the topology of the weak convergence. To this end, we adopt the strategy taken by Berestycki [B]; we fix $A \in \mathcal{B}(B(S-1))$ and look for conditions given in terms of γ, σ, α and the Robin constant $f(\mathbf{x}, r)$ to ensure the $L^2(\mathbb{P})$-convergence of J_ε, which will then be combined with Proposition 4.8 to obtain the $L^1(\mathbb{P})$-convergence of I_ε as well as its uniform integrability. Notice that, by the Fubini theorem,

$$\mathbb{E}[J_\varepsilon^2] = \int\int_{A\times A} \mathbb{E}\left[e^{\widetilde{Y}^{\mathbf{x},\varepsilon,\gamma}+\widetilde{Y}^{\mathbf{y},\varepsilon,\gamma}} 1_{G^{\alpha,\varepsilon_0}_{\mathbf{x},\varepsilon}} 1_{G^{\alpha,\varepsilon_0}_{\mathbf{y},\varepsilon}}\right] \sigma(d\mathbf{x})\sigma(d\mathbf{y}).$$

Proposition 4.10 *Assume that the following two conditions are fulfilled for some $\alpha \in (0, 2\gamma)$:*

$$\lim_{r\downarrow 0} \int_A \exp\left(-\frac{1}{2}(2\gamma-\alpha)^2 f(\mathbf{y}, 7r) + \gamma^2 f(\mathbf{y}, r)\right) \sigma(B(\mathbf{y}, 6r))\sigma(d\mathbf{y}) = 0 \tag{4.36}$$

and

$$\begin{aligned} \lim_{\eta\to 0} \sup_{\varepsilon,\delta\le\eta} \int\int_{A\times A\cap\{|\mathbf{x}-\mathbf{y}|<\eta\}} &\exp\left(-\frac{1}{2}(2\gamma-\alpha)^2 f(\mathbf{y}, |\mathbf{x}-\mathbf{y}| + \varepsilon\vee\delta)\right) \\ &\times \exp\left(\gamma^2\kappa f(\mathbf{y}, (|\mathbf{x}-\mathbf{y}| - (\varepsilon+\delta))\vee(\varepsilon\vee\delta)\right)\sigma(d\mathbf{x})\sigma(d\mathbf{y}) = 0, \end{aligned} \tag{4.37}$$

where κ is a constant appearing in (4.20)*. Then,*

$$\lim_{\eta\downarrow 0} \sup_{\varepsilon,\delta\le\eta} \int\int_{A\times A\cap\{|\mathbf{x}-\mathbf{y}|<\eta\}} \mathbb{E}\left[e^{\widetilde{Y}^{\mathbf{x},\varepsilon,\gamma}+\widetilde{Y}^{\mathbf{y},\delta,\gamma}} 1_{G^{\alpha,\varepsilon_0}_{\mathbf{x},\varepsilon}} 1_{G^{\alpha,\varepsilon_0}_{\mathbf{y},\delta}}\right] \sigma(d\mathbf{x})\sigma(d\mathbf{y}) = 0. \tag{4.38}$$

Proof. Put

$$\begin{cases} \widetilde{G}^{\alpha,\varepsilon_0}_{\mathbf{x},\varepsilon}(\mathbf{y},\delta) = \left\{Y^{\mathbf{x},\overline{r}} \le \alpha f(\mathbf{x},\overline{r}) - \gamma \mathrm{Cov}(Y^{\mathbf{x},\varepsilon} + Y^{\mathbf{y},\delta}, Y^{\mathbf{x},\overline{r}}), \forall r \in (\varepsilon, \varepsilon_0)\right\}, \\ \widetilde{G}^{\alpha,\varepsilon_0}_{\mathbf{y},\delta}(\mathbf{x},\varepsilon) = \left\{Y^{\mathbf{y},\overline{r}} \le \alpha f(\mathbf{y},\overline{r}) - \gamma \mathrm{Cov}(Y^{\mathbf{x},\varepsilon} + Y^{\mathbf{y},\delta}, Y^{\mathbf{y},\overline{r}}), \forall r \in (\delta, \varepsilon_0)\right\}. \end{cases} \tag{4.39}$$

Then, by the Cameron-Martin formula (2.25), we have

$$\begin{aligned} &\int\int_{A\times A\cap\{|\mathbf{x}-\mathbf{y}|<\eta\}} \mathbb{E}\left[e^{\widetilde{Y}^{\mathbf{x},\varepsilon,\gamma}+\widetilde{Y}^{\mathbf{y},\delta,\gamma}} 1_{G^{\alpha,\varepsilon_0}_{\mathbf{x},\varepsilon}} 1_{G^{\alpha,\varepsilon_0}_{\mathbf{y},\delta}}\right] \sigma(d\mathbf{x})\sigma(d\mathbf{y}) \\ &= \int\int_{A\times A\cap\{|\mathbf{x}-\mathbf{y}|<\eta\}} e^{\gamma^2 \mathrm{Cov}(Y^{\mathbf{x},\varepsilon}, Y^{\mathbf{y},\delta})} \mathbb{P}\left(\widetilde{G}^{\alpha,\varepsilon_0}_{\mathbf{x},\varepsilon}(\mathbf{y},\delta) \cap \widetilde{G}^{\alpha,\varepsilon_0}_{\mathbf{y},\delta}(\mathbf{x},\varepsilon)\right) \sigma(d\mathbf{x})\sigma(d\mathbf{y}). \end{aligned} \tag{4.40}$$

We may assume that $\varepsilon \le \delta \le \overline{r}$ because if otherwise we may exchange $(\mathbf{y}, \delta)$ and $(\mathbf{x}, \varepsilon)$. If $0 < \varepsilon \le \delta \le \varepsilon_0$, then $\mathrm{cov}(Y^{\mathbf{x},\varepsilon} + Y^{\mathbf{y},\delta}, Y^{\mathbf{y},\overline{r}}) = f(\mathbf{y}, \overline{r}) + \langle \mu^{\mathbf{x},\varepsilon}, R\mu^{\mathbf{y},\overline{r}}\rangle$ so that

$$\mathbb{P}(\widetilde{G}^{\alpha,\varepsilon_0}_{\mathbf{y},\delta}(\mathbf{x},\varepsilon)) \le \mathbb{P}\left(Y^{\mathbf{y},\overline{r}} \le (\alpha-\gamma) f(\mathbf{y},\overline{r}) - \gamma\langle \mu^{\mathbf{x},\varepsilon}, R\mu^{\mathbf{y},\overline{r}}\rangle\right).$$

Let us make a special choice of r satisfying $f(\mathbf{y}, |\mathbf{x}-\mathbf{y}|+\varepsilon) - 1 < [f(\mathbf{y}, r)] \leq f(\mathbf{y}, |\mathbf{x}-\mathbf{y}|+\varepsilon)$. Then $|\mathbf{x}-\mathbf{y}|+\varepsilon \leq \overline{r}$ and hence $\langle \mu^{\mathbf{x},\varepsilon}, R\mu^{\mathbf{y},\overline{r}}\rangle = f(\mathbf{y}, \overline{r})$. Since $\alpha - 2\gamma < 0$, we get from (4.32),

$$\begin{aligned}
\mathbb{P}\left(\widetilde{G}^{\alpha,\varepsilon_0}_{\mathbf{x},\varepsilon}(\mathbf{y},\delta) \cap \widetilde{G}^{\alpha,\varepsilon_0}_{\mathbf{y},\delta}(\mathbf{x},\varepsilon)\right) &\leq \mathbb{P}\left(\widetilde{G}^{\alpha,\varepsilon_0}_{\mathbf{y},\delta}(\mathbf{x},\varepsilon)\right) \leq \mathbb{P}\left(Y^{\mathbf{y},\overline{r}} \leq (\alpha-2\gamma) f(\mathbf{y},\overline{r})\right) \\
&\leq \exp\left(-\frac{1}{2}(2\gamma-\alpha)^2 (f(\mathbf{y}, |\mathbf{x}-\mathbf{y}|+\varepsilon) - 1)\right).
\end{aligned}$$

If $|\mathbf{x}-\mathbf{y}| \leq 6\delta$, then $f(\mathbf{y}, |\mathbf{x}-\mathbf{y}|+\varepsilon) \geq f(\mathbf{y}, 7\delta)$. As $\mathrm{Cov}(Y^{\mathbf{x},\varepsilon}, Y^{\mathbf{y},\delta}) = \langle \mu^{\mathbf{x},\varepsilon}, R\mu^{\mathbf{y},\delta}\rangle \leq f(\mathbf{y},\delta)$ by (4.16), we have

$$\begin{aligned}
&\int\int_{A\times A\cap\{|\mathbf{x}-\mathbf{y}|\leq 6\delta\}} e^{\gamma^2 \mathrm{Cov}(Y^{\mathbf{x},\varepsilon}, Y^{\mathbf{y},\delta})} \mathbb{P}\left(\widetilde{G}^{\alpha,\varepsilon_0}_{\mathbf{x},\varepsilon}(\mathbf{y},\delta) \cap \widetilde{G}^{\alpha,\varepsilon_0}_{\mathbf{y},\delta}(\mathbf{x},\varepsilon)\right) \sigma(d\mathbf{x})\sigma(d\mathbf{y}) \\
&\leq \int_A \exp\left(-\frac{1}{2}(2\gamma-\alpha)^2 (f(\mathbf{y},7\delta) - 1) + \gamma^2 f(\mathbf{y},\delta)\right) \sigma(B(\mathbf{y},6\delta))\sigma(d\mathbf{y}).
\end{aligned}$$

On the other hand, if $6\delta < |\mathbf{x}-\mathbf{y}|$, then

$$\mathrm{Cov}(Y^{\mathbf{x},\varepsilon}, Y^{\mathbf{y},\delta}) \leq \kappa f(\mathbf{y}, (|\mathbf{x}-\mathbf{y}| - (\varepsilon+\delta)) \vee \delta) + C_1,$$

by (4.20) and accordingly the integral over the domain resricted to $|\mathbf{x}-\mathbf{y}| \geq 6\delta$ of the righthand side of (4.40) is dominated by

$$\begin{aligned}
&\int\int_{A\times A\cap\{6\delta<|\mathbf{x}-\mathbf{y}|<\eta\}} \exp\left(-\frac{1}{2}(2\gamma-\alpha)^2 f(\mathbf{y}, |\mathbf{x}-\mathbf{y}|+\delta) + \gamma^2 \mathrm{Cov}(Y^{\mathbf{x},\varepsilon}, Y^{\mathbf{y},\delta})\right) \sigma(d\mathbf{x})\sigma(d\mathbf{y}) \\
&\leq e^{\gamma^2 C_1} \int\int_{A\times A\cap\{|\mathbf{x}-\mathbf{y}|<\eta\}} \exp\left(-\frac{1}{2}(2\gamma-\alpha)^2 f(\mathbf{y}, |\mathbf{x}-\mathbf{y}|+\delta)\right) \\
&\quad \times \exp\left(\kappa\gamma^2 f(\mathbf{y}, (|\mathbf{x}-\mathbf{y}| - (\varepsilon+\delta)) \vee \delta)\right) \sigma(d\mathbf{x})\sigma(d\mathbf{y}).
\end{aligned}$$

Therefore we have (4.38) under (4.36) and (4.37). □

Proposition 4.11 *Fix $\eta > 0$ and $\alpha > \gamma$. For any Borel subset A of $B(S-1)$,*

$$\begin{aligned}
&\lim_{\varepsilon,\delta\to 0} \int\int_{A\times A\cap\{|\mathbf{x}-\mathbf{y}|\geq\eta\}} \mathbb{E}\left[e^{\widetilde{Y}^{\mathbf{x},\varepsilon,\gamma} + \widetilde{Y}^{\mathbf{y},\delta,\gamma}} 1_{G^{\alpha,\varepsilon_0}_{\mathbf{x},\varepsilon}} 1_{G^{\alpha,\varepsilon_0}_{\mathbf{y},\delta}}\right] \sigma(d\mathbf{x})\sigma(d\mathbf{y}) \\
&= \int\int_{A\times A\cap\{|\mathbf{x}-\mathbf{y}|\geq\eta\}} e^{\gamma^2 r(\mathbf{x},\mathbf{y})} \mathbb{P}\left(\widetilde{G}^{\alpha,\varepsilon_0}_{\mathbf{x},0} \cap \widetilde{G}^{\alpha,\varepsilon_0}_{\mathbf{y},0}\right) \sigma(dx)\sigma(dy) < \infty, \qquad (4.41)
\end{aligned}$$

where

$$\begin{cases}
\widetilde{G}^{\alpha,\varepsilon_0}_{\mathbf{x},0} = \{Y^{\mathbf{x},\overline{r}} \leq (\alpha-\gamma) f(\mathbf{x},\overline{r}) - \gamma \widehat{R}\mu^{\mathbf{x},\overline{r}}(\mathbf{y}),\ \forall r \in (0,\varepsilon_0)\} \\
\widetilde{G}^{\alpha,\varepsilon_0}_{\mathbf{y},0} = \{Y^{\mathbf{y},\overline{r}} \leq (\alpha-\gamma) f(\mathbf{y},\overline{r}) - \gamma \widehat{R}\mu^{\mathbf{y},\overline{r}}(\mathbf{x}),\ \forall r \in (0,\varepsilon_0)\}
\end{cases} \qquad (4.42)$$

Proof. Similarly to (4.40), we have

$$\begin{aligned}
&\int\int_{A\times A\cap\{|\mathbf{x}-\mathbf{y}|\geq\eta\}} \mathbb{E}\left[e^{\widetilde{Y}^{\mathbf{x},\varepsilon,\gamma} + \widetilde{Y}^{\mathbf{y},\delta,\gamma}} 1_{G^{\alpha,\varepsilon_0}_{\mathbf{x},\varepsilon}} 1_{G^{\alpha,\varepsilon_0}_{\mathbf{y},\delta}}\right] \sigma(d\mathbf{x})\sigma(d\mathbf{y}) \\
&= \int\int_{A\times A\cap\{|\mathbf{x}-\mathbf{y}|\geq\eta\}} \mathbb{E}\left[e^{\gamma^2 \mathrm{Cov}(Y^{\mathbf{x},\varepsilon}, Y^{\mathbf{y},\delta})} 1_{\widetilde{G}^{\alpha,\varepsilon_0}_{\mathbf{x},\varepsilon}(\mathbf{y},\delta) \cap \widetilde{G}^{\alpha,\varepsilon_0}_{\mathbf{y},\delta}(\mathbf{x},\varepsilon)}\right] \sigma(dx)\sigma(dy) \\
&= \int\int_{A\times A\cap\{|\mathbf{x}-\mathbf{y}|\geq\eta\}} e^{\gamma^2 \langle \mu^{\mathbf{x},\varepsilon}, R\mu^{\mathbf{y},\delta}\rangle} \mathbb{P}\left(\widetilde{G}^{\alpha,\varepsilon_0}_{\mathbf{x},\varepsilon}(\mathbf{y},\delta) \cap \widetilde{G}^{\alpha,\varepsilon_0}_{\mathbf{y},\delta}(\mathbf{x},\varepsilon)\right) \sigma(d\mathbf{x})\sigma(d\mathbf{y}). \qquad (4.43)
\end{aligned}$$

By rewriting (4.39), we also have

$$\begin{cases} \widetilde{G}^{\alpha,\varepsilon_0}_{\mathbf{x},\varepsilon}(\mathbf{y},\delta) = \{Y^{\mathbf{x},\overline{r}} \le (\alpha-\gamma)f(\mathbf{x},\overline{r}) - \gamma\langle \mu^{\mathbf{y},\delta}, R\mu^{\mathbf{x},\overline{r}}\rangle,\ \forall r\in(\varepsilon,\varepsilon_0)\}, \\ \widetilde{G}^{\alpha,\varepsilon_0}_{\mathbf{y},\delta}(\mathbf{x},\varepsilon) = \{Y^{\mathbf{y},\overline{r}} \le (\alpha-\gamma)f(\mathbf{y},\overline{r}) - \gamma\langle \mu^{\mathbf{x},\varepsilon}, R\mu^{\mathbf{y},\overline{r}}\rangle,\ \forall r\in(\delta,\varepsilon_0)\}. \end{cases} \tag{4.44}$$

Since $\widehat{R}\mu^{\mathbf{x},\overline{r}}(\mathbf{z})$ is continuous on $B(S)$ by Lemma 4.4,

$$\lim_{\delta\downarrow 0}\langle \mu^{\mathbf{y},\delta}, \widehat{R}\mu^{\mathbf{x},\overline{r}}\rangle = \widehat{R}\mu^{\mathbf{x},\overline{r}}(\mathbf{y}), \quad \mathbf{y}\in B(S-1). \tag{4.45}$$

We now let $v(\alpha,\varepsilon_0,\mathbf{x},\mathbf{y},\varepsilon,\delta) = \mathbb{P}\left(\widetilde{G}^{\alpha,\varepsilon_0}_{\mathbf{x},\varepsilon}(\mathbf{y},\delta)\cap \widetilde{G}^{\alpha,\varepsilon_0}_{\mathbf{y},\delta}(\mathbf{x},\varepsilon)\right)$. It follows from (4.45) and the continuity of the finite dimensional Gaussian distribution function that

$$\begin{aligned} &\limsup_{\varepsilon\downarrow 0,\ \delta\downarrow 0}\ v(\alpha,\varepsilon_0,\mathbf{x},\mathbf{y},\varepsilon,\delta) \\ &\le \mathbb{P}\Big(\{Y^{\mathbf{x},\overline{r}} \le (\alpha-\gamma)f(\mathbf{x},\overline{r}) - \gamma\widehat{R}\mu^{\mathbf{x},\overline{r}}(\mathbf{y}),\ \forall r\in(\varepsilon_1,\varepsilon_0)\} \\ &\cap\{Y^{\mathbf{y},\overline{r}} \le (\alpha-\gamma)f(\mathbf{y},\overline{r}) - \gamma\widehat{R}\mu^{\mathbf{y},\overline{r}}(\mathbf{x}),\ \forall r\in(\delta_1,\varepsilon_0)\}\Big), \end{aligned}$$

for arbitrarily fixed $\varepsilon_1<\varepsilon_0,\ \delta_1<\varepsilon_0$. We then let $\delta_1\downarrow 0,\ \varepsilon_1\downarrow 0$ to obtain

$$\limsup_{\varepsilon\downarrow 0,\ \delta\downarrow 0}\ v(\alpha,\varepsilon_0,\mathbf{x},\mathbf{y},\varepsilon,\delta) \le \mathbb{P}(\widetilde{G}^{\alpha,\varepsilon_0}_{\mathbf{x},0}\cap \widetilde{G}^{\alpha,\varepsilon_0}_{\mathbf{y},0}). \tag{4.46}$$

Fix any $\eta>0$. By taking $r_0=\frac{\eta}{2}\wedge\frac14$ in Lemma 4.6, we find, for $\mathbf{x},\mathbf{y}\in B(S-1)$ with $|\mathbf{x}-\mathbf{y}|>\eta$, a constant $M_2>0$ depending only on η such that $|\langle \mu^{\mathbf{y},\delta}, \widehat{R}\mu^{\mathbf{x},\overline{r}}\rangle| \le M_2$ for any $\delta\in(0,\eta/2)$ and $\overline{r}\in(0,\frac{\eta}{4}\wedge\frac18)$. According to the fine properties of the function $f(\mathbf{x},r)$ in r stated in the first part of this subsection, we see that $\varepsilon_1\downarrow 0$ implies $\overline{\varepsilon}_1 = f^{-1}(\mathbf{x},[f(\mathbf{x}.\varepsilon_1)])\downarrow 0$. Hence one can choose $\varepsilon_1>0$ with $\overline{\varepsilon}_1<\frac{\eta}{4}\wedge\frac18$ so that $\overline{r}<\frac{\eta}{4}\wedge\frac18$ whenever $r<\varepsilon_1$. Further, if we let $D(\mathbf{x},\varepsilon_1)=\{[f(\mathbf{x},r)]: r\in(0,\varepsilon_1)\}(\subset\mathbb{N})$, then $D(\mathbf{x},\varepsilon_1)=\{k\in\mathbb{N}: k\ge[f(\mathbf{x},\varepsilon_1)]\}$.

By using the tail distribution estimate (4.32), we therefore have for any $\delta\in(0,\eta/2)$

$$\begin{aligned} \mathbb{P}(\widetilde{G}^{\alpha,\varepsilon_1}_{\mathbf{x},\varepsilon}(\mathbf{y},\delta)^c) &\le P\left(\bigcup_{\overline{r}\in f^{-1}(\mathbf{x},D(\mathbf{x},\varepsilon_1))}\{Y^{\mathbf{x},\overline{r}} > (\alpha-\gamma)f(\mathbf{x},\overline{r}) - \gamma\langle \mu^{\mathbf{y},\delta}, \widehat{R}\mu^{\mathbf{x},\overline{r}}\rangle\}\right) \\ &\le \sum_{\overline{r}\in f^{-1}(\mathbf{x},D(\mathbf{x},\varepsilon_1))}\mathbb{P}\left(Y^{\mathbf{x},\overline{r}} > (\alpha-\gamma)f(\mathbf{x},\overline{r}) - \gamma\langle \mu^{\mathbf{y},\delta}, \widehat{R}\mu^{\mathbf{x},\overline{r}}\rangle\right) \\ &\le \widetilde{M}\sum_{\overline{r}\in f^{-1}(\mathbf{x},D(\mathbf{x},\varepsilon_1))}\exp\left[-\frac{(\alpha-\gamma)^2}{2}f(\mathbf{x},\overline{r})\right] = \widetilde{M}\sum_{k\ge[f(\mathbf{x},\varepsilon_1)]}\exp\left[-\frac{(\alpha-\gamma)^2}{2}k\right], \end{aligned}$$

where $\widetilde{M}=\exp[(\alpha-\gamma)\gamma M_2]$. Hence $\lim_{\varepsilon_1\downarrow 0}\mathbb{P}(\widetilde{G}^{\alpha,\varepsilon_1}_{\mathbf{x},\varepsilon}(\mathbf{y},\delta)^c)=0$ uniformly in $\delta\in(0,\eta/2)$. Similarly $\lim_{\delta_1\downarrow 0}\mathbb{P}(\widetilde{G}^{\alpha,\delta_1}_{\mathbf{y},\delta}(\mathbf{x},\varepsilon)^c)=0$ uniformly in $\varepsilon\in(0,\eta/2)$.

We just saw that, for any small $a>0$, there is $b>0$ such that $\mathbb{P}(\widetilde{G}^{\alpha,\varepsilon_1}_{\mathbf{x},\varepsilon}(\mathbf{y},\delta)^c)<a$ for any $\varepsilon_1<b$ uniformly in $\delta\in(0,\eta/2)$, and $\mathbb{P}(\widetilde{G}^{\alpha,\delta_1}_{\mathbf{y},\delta}(\mathbf{x},\varepsilon)^c)<a$ for any $\delta_1<b$ uniformly in $\varepsilon\in(0,\eta/2)$.

Let $A_\eta=\{(\mathbf{x},\mathbf{y})\in B(S-1)^2: |\mathbf{x}-\mathbf{y}|>\eta\}$. It follows from $v(\alpha,\varepsilon_0,\mathbf{x},\mathbf{y},\varepsilon,\delta)\ge v(\alpha,\varepsilon_0,\mathbf{x},\mathbf{y},\varepsilon_1,\delta_1) - \mathbb{P}(\widetilde{G}^{\alpha,\varepsilon_1}_{\mathbf{x},\varepsilon}(\mathbf{y},\delta)^c) - \mathbb{P}(\widetilde{G}^{\alpha,\delta_1}_{\mathbf{y},\delta}(\mathbf{x},\varepsilon)^c)$ that

$$\begin{aligned} &\liminf_{\varepsilon\downarrow 0,\ \delta\downarrow 0}\ v(\alpha,\varepsilon_0,\mathbf{x},\mathbf{y},\varepsilon,\delta)1_{A_\eta}((\mathbf{x},\mathbf{y})) \\ &\ge \mathbb{P}\Big(\{Y^{\mathbf{x},\overline{r}} \le (\alpha-\gamma)f(\mathbf{x},\overline{r}) - \gamma\widehat{R}\mu^{\mathbf{x},\overline{r}}(\mathbf{y}),\ \forall r\in(\varepsilon_1,\varepsilon_0)\} \\ &\quad\cap\{Y^{\mathbf{y},\overline{r}} \le (\alpha-\gamma)f(\mathbf{y},\overline{r}) - \gamma\widehat{R}\mu^{\mathbf{y},\overline{r}}(\mathbf{x}),\ \forall r\in(\delta_1,\varepsilon_0)\}\Big)1_{A_\eta}((\mathbf{x},\mathbf{y})) - 2a, \end{aligned}$$

for any $\varepsilon_1 < b,\ \delta_1 < b$. By letting $\varepsilon_1 \downarrow 0, \delta_1 \downarrow 0$ and then $a \downarrow 0$, we arrive at

$$\liminf_{\varepsilon\downarrow 0,\ \delta\downarrow 0} v(\alpha,\varepsilon_0,\mathbf{x},\mathbf{y},\varepsilon,\delta)1_{A_\eta}((\mathbf{x},\mathbf{y})) \geq \mathbb{P}(\widetilde{G}^{\alpha,\varepsilon_0}_{\mathbf{x},0} \cap \widetilde{G}^{\alpha,\varepsilon_0}_{\mathbf{y},0})1_{A_\eta}((\mathbf{x},\mathbf{y})). \tag{4.47}$$

On the set $(B(S-1)\times B(S-1))\cap\{|\mathbf{x}-\mathbf{y}| \geq \eta\}$, $\langle \mu^{\mathbf{x},\varepsilon}, R\mu^{\mathbf{y},\delta}\rangle$ is uniformly bounded and converges to $r(\mathbf{x},\mathbf{y})$ as $\varepsilon \downarrow 0, \delta \downarrow 0$ a.e. $\sigma\times\sigma$ by Proposition 4.7. Further $\sigma\times\sigma(B(S-1)\times B(S-1)) < \infty$. Therefore we obtain (4.41) from (4.43), (4.46) and (4.47). □

Recall the random variables $I_\varepsilon(\omega)$ and $J_\varepsilon(\omega)$ defined by (4.35).

Theorem 4.12 *Assume that the conditions* (4.36), (4.37) *in* Proposition 4.10 *are fulfilled for some* $\alpha \in (\gamma, 2\gamma)$. *Then* $J_\varepsilon(\omega)$ *converges in* $L^2(\Omega;\mathbb{P})$ *as* $\varepsilon \to 0$. *Furthermore,* $I_\varepsilon(\omega)$ *is uniformly integrable with respect to* $0 < \varepsilon < 1$ *and* $A \in \mathcal{B}(B(S-1))$, *and it converges in* $L^1(\Omega,\mathbb{P})$ *as* $\varepsilon \to 0$.

Proof. For any $\varepsilon, \delta \in (0,\varepsilon_0)$,

$$\begin{aligned}\mathbb{E}\left[|J_\varepsilon(\omega) - J_\delta(\omega)|^2\right] &= \int\!\!\int_{A\times A} \mathbb{E}\left[e^{\widetilde{Y}^{\mathbf{x},\varepsilon,\gamma}+\widetilde{Y}^{\mathbf{y},\varepsilon,\gamma}} 1_{G^{\alpha,\varepsilon_0}_{\mathbf{x},\varepsilon}} 1_{G^{\alpha,\varepsilon_0}_{\mathbf{y},\varepsilon}}\right]\sigma(d\mathbf{x})\sigma(d\mathbf{y}) \\ &\quad + \int\!\!\int_{A\times A} \mathbb{E}\left[e^{\widetilde{Y}^{\mathbf{x},\delta,\gamma}+\widetilde{Y}^{\mathbf{y},\delta,\gamma}} 1_{G^{\alpha,\varepsilon_0}_{\mathbf{x},\delta}} 1_{G^{\alpha,\varepsilon_0}_{\mathbf{y},\delta}}\right]\sigma(d\mathbf{x})\sigma(d\mathbf{y}) \\ &\quad -2\int\!\!\int_{A\times A} \mathbb{E}\left[e^{\widetilde{Y}^{\mathbf{x},\varepsilon,\gamma}+\widetilde{Y}^{\mathbf{y},\delta,\gamma}} 1_{G^{\alpha,\varepsilon_0}_{\mathbf{x},\varepsilon}} 1_{G^{\alpha,\varepsilon_0}_{\mathbf{y},\delta}}\right]\sigma(d\mathbf{x})\sigma(d\mathbf{y}).\end{aligned}$$

Express each of three integrals on $A\times A$ in the righthand side as a sum of integrals over $(A\times A)\cap\{|\mathbf{x}-\mathbf{y}| < \eta\}$ and $(A\times A)\cap\{|\mathbf{x}-\mathbf{y}| \geq \eta\}$. For an arbitrarily small $a > 0$, there exists $\eta_a > 0$ such that each integral over $(A\times A)\cap\{|\mathbf{x}-\mathbf{y}| \leq \eta\}$ with $\eta = \eta_a$ is smaller than a uniformly in $\varepsilon > 0, \delta > 0$ by virtue of Proposition 4.10. On the other hand, the limits of the integrals over $(A\times A)\cap\{|\mathbf{x}-\mathbf{y}| \geq \eta_a\}$ as $\varepsilon,\delta \to 0$ exist and cancel each others by Proposition 4.11, resulting in $\limsup_{\varepsilon,\ \delta\to 0}\mathbb{E}\left[|J_\varepsilon(\omega) - J_\delta(\omega)|^2\right] \leq 4a$. Since a is arbitrary, we obtain the L^2-convergence of $J_\varepsilon(\omega)$.

Proposition 4.8 says that, by taking small ε_0, $\mathbb{E}(e^{\widetilde{Y}^{\mathbf{x},\varepsilon,\gamma}} 1_{\Omega\setminus G^{\alpha,\varepsilon_0}_{\mathbf{x},\varepsilon}})$ becomes arbitrarily small uniformly in $0 < \varepsilon < \varepsilon_0$ and $\mathbf{x} \in B(S-1)$. Since $\mathbb{E}[(J_\varepsilon)^2]$ is uniformly bounded relative to ε, $\{J_\varepsilon\}$ is uniformly integrable. Hence $\{I_\varepsilon\}$ is also uniformly integrable. As J_ε converges in $L^2(\Omega,\mathbb{P})$, it also converges in $L^1(\Omega,\mathbb{P})$. Noting that $\mathbb{E}[I_\varepsilon - J_\varepsilon]$ is small uniformly relative to $\varepsilon < \varepsilon_0$ by taking small ε_0, we also see that I_ε converges in $L^1(\Omega,\mathbb{P})$. □

In formulating the following theorem, we put $D = B(S-1)$. Consider the family $\mathcal{M}(D)$ of all finite positive measures on $(D,\mathcal{B}(D))$ equipped with the topology of weak convergence: $\mu_n(D) \in \mathcal{M}(D)$ converges to $\mu \in \mathcal{M}(D)$ as $n \to \infty$ if $\lim_{n\to\infty}\langle f,\mu_n\rangle = \langle f,\mu\rangle$ for any $f \in C_b(D)$, where $\langle f,\mu\rangle = \int_D f(x)\mu(dx)$. This topology is induced by the metric ρ on $\mathcal{M}(D)$ defined by

$$\rho(\mu,\nu) = \sum_{n=1}^{\infty} 2^{-n}\left(|\langle g_n,\mu\rangle - \langle g_n,\nu\rangle| \wedge 1\right), \quad \mu,\nu \in \mathcal{M}(D), \tag{4.48}$$

where $\{g_n\}$ is a countable dense subfamily of $C(\overline{D})$ (cf. [IW, Prop.2.5]).

For each set $A \in \mathcal{B}(D)$, the integral $\int_A e^{\widetilde{Y}^{\mathbf{x},\varepsilon}(\omega)}\sigma(d\mathbf{x})$ will be denoted by $\mu_\varepsilon(A,\omega)$ instead of $I_\varepsilon(\omega)$. Notice that $\mu_\varepsilon(\cdot,\omega) \in \mathcal{M}(D)$ a.s.

Theorem 4.13 *Assume that the conditions* (4.36), (4.37) *in* Proposition 4.10 *are fulfilled for some* $\alpha \in (\gamma, 2\gamma)$. *Then there exists* $\overline{\mu}(\cdot, \omega) \in \mathcal{M}(D)$ *uniquely a.s. such that* $\mu_\varepsilon(\cdot, \omega)$ *converges in probability to* $\overline{\mu}(\cdot, \omega)$ *as* $\varepsilon \downarrow 0$ *relative to the metric* ρ *on* $\mathcal{M}(D)$*: for any* $\delta > 0$,

$$\lim_{\varepsilon \downarrow 0} \mathbb{P}\left(\rho(\mu_\varepsilon, \overline{\mu}) > \delta\right) = 0. \tag{4.49}$$

Proof. Denote by $\mathcal{A}$ the family of rectangles in D of the form $[r_1, s_1) \times [r_2, s_2)$ with rational numbers $r_1 < s_1$ and $r_2 < s_2$. We also put $D_k = B(S - 1 - 1/k),\ k \geq 1$. By virtue of Theorem 4.12, for each $A \in \mathcal{B}(D)$, $\mu_\varepsilon(A, \omega)$ converges in probability as $\varepsilon \downarrow 0$ to a random variable which we denote by $\mu(A, \omega)$.

First we will prove that there exists a family of random variables $\{\overline{\mu}(A, \omega);\ A \in \mathcal{B}(D)\}$ such that $\overline{\mu}(\cdot, \omega) \in \mathcal{M}(D)$ for almost all $\omega \in \Omega$ and

$$\overline{\mu}(A, \omega) = \mu(A, \omega) \quad \text{for any } A \in \mathcal{A}, \quad \text{a.s.} \tag{4.50}$$

Further we will prove that any sequence $\varepsilon_n \downarrow 0$ admits a subsequence $\{\varepsilon'_n\}$ such that $\mu_{\varepsilon'_n}(\cdot, \omega)$ is weakly convergent to $\overline{\mu}(\cdot, \omega)$ as $n \to \infty$ a.s.

Take any sequence $\varepsilon_n \downarrow 0$. As the family $\mathcal{A}$ is countable, there exists its subsequence $\{\varepsilon'_n\}$ such that $\lim_{n\to\infty} \mu_{\varepsilon'_n}(D, \omega) = \mu(D, \omega)$, $\lim_{n\to\infty} \mu_{\varepsilon'_n}(D_k, \omega) = \mu(D_k, \omega)$ for all $k \geq 1$ and $\lim_{n\to\infty} \mu_{\varepsilon'_n}(A, \omega) = \mu(A, \omega)$ for all $A \in \mathcal{A}$ a.s. say for all $\omega \in \Omega' \subset \Omega$ with $\mathbb{P}(\Omega') = 1$. Since $\lim_{n\to\infty} \mu_{\varepsilon'_n}(D, \omega) = \mu(D, \omega)$ for all $\omega \in \Omega'$ and $\{\mu_\varepsilon(D); \varepsilon > 0\}$ is uniformly integrable by Theorem 4.12,

$$\mathbb{E}[\mu(D, \omega)] = \lim_{n\to\infty} \mathbb{E}[\mu_{\varepsilon'_n}(D, \omega)] = \sigma(D) < \infty.$$

Hence $\mu(D, \omega) < \infty$ a.s. Similarly, since $\mu(D_k, \omega)$ is non-decreasing relative to k and

$$\lim_{k\to\infty} \mathbb{E}[\mu(D, \omega) - \mu(D_k, \omega)] = \lim_{k\to\infty} \lim_{n\to\infty} \mathbb{E}[\mu_{\varepsilon'_n}(D, \omega) - \mu_{\varepsilon'_n}(D_k, \omega)] = \lim_{k\to\infty} \sigma(D \setminus D_k) = 0,$$

it follows that $\lim_{k\to\infty} \mu(D_k, \omega) = \mu(D, \omega)$ a.s. Therefore, we may and shall assume that $\mu(D, \omega) < \infty$ and $\lim_{k\to\infty} \mu(D_k, \omega) = \mu(D, \omega)$ for all $\omega \in \Omega'$.

For any $A \in \mathcal{A}$, let A° and $\overline{A}$ be the interior and the closure of A, respectively. Since $\mu_\varepsilon(A^\circ, \omega) \leq \mu_\varepsilon(A, \omega) \leq \mu_\varepsilon(\overline{A}, \omega)$ and $\mathbb{E}[\mu_\varepsilon(A^\circ)] = \sigma(A) = \mathbb{E}[\mu_\varepsilon(\overline{A})]$ by the stated assumption that $\sigma(\partial A) = 0$, $\mathbb{E}[\mu(A^\circ)] = \mathbb{E}[\mu(\overline{A})]$. Hence $\mu(A^\circ, \omega) = \mu(A, \omega) = \mu(\overline{A}, \omega)$ for almost all ω and we may assume that this holds for all $\omega \in \Omega'$.

Fix $\omega \in \Omega'$. For any $\delta > 0$, take a number k_0 such that $\mu(D, \omega) - \mu(D_k, \omega) < \delta\mu(D, \omega)/2$ for any $k \geq k_0$. Further, there exists n_0 such that $|\mu_{\varepsilon'_n}(D \setminus D_{k_0}, \omega) - (\mu(D, \omega) - \mu(D_{k_0}, \omega))| < \delta\mu(D, \omega)/2$ for any $n \geq n_0$. Therefore $\mu_{\varepsilon'_n}(D \setminus D_k) \leq \mu_{\varepsilon'_n}(D \setminus D_{k_0}) < \delta\mu(D, \omega)$ for any $n \geq n_0$ and $k \geq k_0$. By taking large k_0 if necessary, this holds for all $n \geq 1$ and $k \geq k_0$. Since $D_k \subset \overline{D}_k \subset D$ and $\overline{D}_k$ is a compact subset of D, this means the uniform tightness of $\{\mu_{\varepsilon'_n}(\cdot, \omega)\}$. As a consequence, any subsequence of $\{\varepsilon'_n\}$ admits a further subsequence $\{\varepsilon''_n\}$ such that $\{\mu_{\varepsilon''_n}(\cdot, \omega)\}$ converges weakly to some measure $\overline{\mu}(\cdot, \omega)$.

For any $A \in \mathcal{A}$, choose $C_k, B_k \in \mathcal{A},\ k \geq 1$, such that $\overline{C}_k$ increases to A° and B_k° decreases to $\overline{A}$ as $k \to \infty$. Then

$$\mu(\overline{C}_k) = \mu(C_k) = \lim_n \mu_{\varepsilon'_n}(C_k) \leq \limsup_n \mu_{\varepsilon''_n}(\overline{C}_k) \leq \overline{\mu}(\overline{C}_k) \leq \overline{\mu}(A^\circ) \leq \overline{\mu}(A),$$

$$\mu(B_k^\circ) = \mu(B_k) = \lim_n \mu_{\varepsilon'_n}(B_k) \geq \liminf_n \mu_{\varepsilon''_n}(B_k^\circ) \geq \overline{\mu}(B_k^\circ) \geq \overline{\mu}(\overline{A}) \geq \overline{\mu}(A),$$

and so

$$\mu(\overline{C}_k) \leq \overline{\mu}(A^\circ) \leq \overline{\mu}(A) \leq \overline{\mu}(\overline{A}) \leq \mu(B_k^\circ), \quad k \geq 1. \tag{4.51}$$

It also follows from

$$\mathbb{E}[\lim_k \mu(\overline{C}_k)] = \lim_k \sigma(\overline{C}_k) = \sigma(A^\circ) = \mathbb{E}[\mu(A)] = \sigma(\overline{A}) = \lim_k \sigma(B_k^\circ) = \mathbb{E}[\lim_k \mu(B_k^\circ)],$$

that

$$\lim_k \mu(\overline{C}_k) = \mu(A) = \lim_k \mu(B_k^\circ), \quad \text{a.s.},$$

which combined with (4.51) yields (4.50).

Since this holds for any subsequence of $\{\varepsilon'_n\}$ and $\mathcal{A}$ generates the Borel σ-field $\mathcal{B}(D)$, $\mu_{\varepsilon'_n}$ converges weakly to $\overline{\mu}$ a.s. Actually every $A \in \mathcal{A}$ is a $\overline{\mu}$-continuity set a.s.

We have seen that any sequence $\varepsilon_n \downarrow 0$ admits a subsequence $\{\varepsilon'_n\}$ such that $\lim_{n\to\infty} \rho(\mu_{\varepsilon'_n}, \overline{\mu}) = 0$ a.s. and consequently $\lim_{n\to\infty} \mathbb{P}\left(\rho(\mu_{\varepsilon'_n}, \overline{\mu}) > \delta\right) = 0$ for any $\delta > 0$. This means (4.49). □

We call $\{\overline{\mu}(A,\omega); A \in \mathcal{B}(D)\}$ in the above theorem the *Gaussian multiplicative chaos* associated with the Gaussian field $\mathbb{G}(\mathcal{E})$. We notice that, due to the uniform integrability of I_ε in Theorem 4.12 and (4.50), $\mathbb{E}[\overline{\mu}(A)] = \sigma(A)$ for any $A \in \mathcal{A}$, so that $\overline{\mu}$ is no-trivial if and only if so is σ. We further notice that the validity of the conditions (4.36), (4.37) in Proposition 4.10 depends on the choice of the measure σ and the value $\kappa > 0$ in (4.20). In examples in the next section, we examine the possible range of the value γ to ensure the validity of these conditions.

5 Examples of Gaussian multiplicative chaos for recurrent forms

Example 5.1 We consider the case that $E = \mathbb{C}$, m is the Lebesgue measure on $\mathbb{C}$ and $(\mathcal{E}, \mathcal{F})$ is the regular recurrent Dirichlet form $(\frac{1}{2}\mathbf{D}_{\mathbb{C}}, H^1(\mathbb{C}))$ on $L^2(\mathbb{C}) = L^2(\mathbb{C}; m)$, where

$$\mathbf{D}_{\mathbb{C}}(u,v) = \int_{\mathbb{C}} \nabla u(\mathbf{x}) \cdot \nabla v(\mathbf{x}) d\mathbf{x}, \quad H^1(\mathbb{C}) = \{u \in L^2(\mathbb{C}); |\nabla u| \in L^2(\mathbb{C})\}.$$

The associated diffusion $\mathbb{M} = (X_t, \mathbb{P}_x)$ is the planar Brownian motion.

Clearly the conditions **(B.1)** and **(B.2)** are satisfied. We take

$$F = \overline{B(S+1)} \setminus B(S)$$

as an admissible set and let $\{R\mu : \mu \in \overline{\mathcal{M}_0(\mathbb{C})}\}$ be the family of recurrent potentials relative to F and $\mu^{\mathbf{x},r}$ be the equilibrium measure for $\overline{B(\mathbf{x},r)} (\subset B(S))$ relatove to F. The uniform probability measure on $\partial B(\mathbf{x},r)$ will be denoted by $\nu^{\mathbf{x},r}$. The logarithmic potential $U\mu$ of a measure $\mu \in \overset{\circ}{\mathcal{M}}_0(\mathbb{C})$ on $\mathbb{C}$ is defined in §2.4 (III). $\widetilde{m}_F$ will designate the probability measure defined by (3.15).

It holds then that

$$\mu^{\mathbf{x},r} = \nu^{\mathbf{x},r}, \tag{5.1}$$

and, for the version $\widehat{R}\mu^{\mathbf{x},r}$ of $R\mu^{\mathbf{x},r}$ introduced in Lemma 4.4,

$$\widehat{R}\mu^{\mathbf{x},r}(\mathbf{z}) = \frac{1}{\pi} \log \frac{1}{|\mathbf{x} - \mathbf{z}| \vee r} - 2\ell(S) + \langle \widetilde{m}_F, U\widetilde{m}_F \rangle, \quad \text{for every } \mathbf{z} \in B(S), \tag{5.2}$$

where $\ell(S)$ is a constant defined by

$$\ell(S) = -\frac{S^2}{\pi(2S+1)} \log(1 + \frac{1}{S}) - \frac{1}{\pi}[\log(S+1) - 1/2]. \tag{5.3}$$

A proof will be given in the last part of this example. (5.2) means that the Robin constant for $B(\mathbf{x}, r)$ equals

$$f(\mathbf{x}, r) = \frac{1}{\pi} \log \frac{1}{r} - 2\ell(S) + \langle \widetilde{m}_F, U\widetilde{m}_F \rangle, \tag{5.4}$$

which is independent of $\mathbf{x} \in B(S)$, and consequently the condition **(B.3)** is trivially fulfilled. Further (5.2) implies (4.19) with $\kappa = 1$ so that (4.20) with $\kappa = 1$ is fulfilled by Lemma 4.5.

The extended Dirichlet space $\mathcal{F}_e$ is now the Beppo Levi space $\mathrm{BL}(\mathbb{C})$ as was mentioned in §2.4 (III). Let $\{X_u; u \in \mathrm{BL}(\mathbb{C})\}$ be the centered Gaussian field defined on a probability space $(\Omega, \mathcal{B}, \mathbb{P})$ with covariance $\mathbb{E}[X_u X_v] = \frac{1}{2}\mathbf{D}_{\mathbb{C}}(u, v)$ $u, v \in \mathrm{BL}(\mathbb{C})$. Define $\widetilde{Y}^{\mathbf{x},\varepsilon,\gamma}$ by (4.31). We now check for what $\gamma > 0$ the conditions (4.36) and (4.37) with $\kappa = 1$ are satisfied to ensure the convergence in probability of random measures $\mu_\varepsilon(A, \omega) = \int_A e^{\widetilde{Y}^{\mathbf{x},\varepsilon,\gamma}} \sigma(d\mathbf{x})$, $A \in \mathcal{B}(B(S-1))$ as $\varepsilon \downarrow 0$ toward a non-trivial random measure on $B(S-1)$. Let $\sigma(d\mathbf{x})$ be a non-trivial positive finite measure on $B(S-1)$ absolutely continuous with respect to the Lebesgue measure satisfying

$$\int_{B(S-1)} \sigma(B(\mathbf{y}, r))\sigma(d\mathbf{y}) \le C_3 r^2, \quad \text{for some constant } C_3 > 0, \tag{5.5}$$

$$\int\int_{B(S-1)\times B(S-1)} \frac{1}{|\mathbf{x}-\mathbf{y}|^{2-\widetilde{\varepsilon}}} \sigma(d\mathbf{x})\sigma(d\mathbf{y}) < \infty, \quad \text{for any } \widetilde{\varepsilon} > 0. \tag{5.6}$$

σ fulfills (5.5) and (5.6) if its density function with respect to the Lebesgue measure is bounded.

For a given $\gamma > 0$, we choose α such that

$$\gamma < \alpha < 2\gamma, \qquad \frac{1}{2\pi}(2\gamma - \alpha)^2 - \frac{\gamma^2}{\pi} + 2 > 0. \tag{5.7}$$

We can actually find α sufficiently close to γ and satisfying the property (5.7) provided that

$$\gamma < 2\sqrt{\pi}. \tag{5.8}$$

In this section, $K_1 \sim K_9$ will denote some positive constants. By virtue of the simple expression (5.4) of the Robin constant $f(\mathbf{x}, r)$, the integral in (4.36) is, under the assumption (5.5), dominated by $K_1 r^{\frac{(2\gamma-\alpha)^2}{2\pi} - \frac{\gamma^2}{\pi} + 2}$, which tends to 0 as $r \downarrow 0$ in view of the property (5.7), yielding (4.36).

Substituting (5.4) into the integral of the left hand side of (4.37) with $\kappa = 1$ by assuming that $\varepsilon \le \delta$, we see that the integral equals

$$\begin{aligned}
&K_2 \int\int_{A\times A\cap\{|\mathbf{x}-\mathbf{y}|<\eta\}} \exp\left(\frac{1}{2\pi}(2\gamma-\alpha)^2 \log(|\mathbf{x}-\mathbf{y}|+\delta)\right) \\
&\qquad \times \exp\left(-\frac{\gamma^2}{\pi}\log((|\mathbf{x}-\mathbf{y}| - (\varepsilon+\delta)) \vee \delta)\right)\sigma(d\mathbf{x})\sigma(d\mathbf{y}) \\
&\le K_2 \int\int_{A\times A\cap\{|\mathbf{x}-\mathbf{y}|<2(\varepsilon+\delta)\}} (|\mathbf{x}-\mathbf{y}|+\delta)^{\frac{1}{2\pi}(2\gamma-\alpha)^2} \delta^{-\frac{\gamma^2}{\pi}} \sigma(d\mathbf{x})\sigma(d\mathbf{y}) \\
&\quad + K_2 \int\int_{A\times A\cap\{2(\varepsilon+\delta)\le|\mathbf{x}-\mathbf{y}|<\eta\}} (|\mathbf{x}-\mathbf{y}|+\delta)^{\frac{1}{2\pi}(2\gamma-\alpha)^2} (|\mathbf{x}-\mathbf{y}| - (\varepsilon+\delta))^{-\frac{\gamma^2}{\pi}} \sigma(d\mathbf{x})\sigma(d\mathbf{y}) \\
&= \mathrm{I} + \mathrm{II}
\end{aligned}$$

Taking $\widetilde{\varepsilon} \in (0, 2)$ with $\frac{1}{2\pi}(2\gamma-\alpha)^2 - \frac{\gamma^2}{\pi} + 2 - \widetilde{\varepsilon} > 0$, we let

$$K_3 = \int\int_{B(S-1)\times B(S-1)} 1/|\mathbf{x}-\mathbf{y}|^{2-\widetilde{\varepsilon}} \sigma(d\mathbf{x})\sigma(d\mathbf{y})$$

which is finite by (5.6). Since $|\mathbf{x}-\mathbf{y}| \le 2(\varepsilon+\delta) \le 4\delta$ in the integrand of I,

$$
\begin{aligned}
\mathrm{I} &= K_2 \int\int_{A\times A\cap\{|\mathbf{x}-\mathbf{y}|<\varepsilon+\delta\}} (|\mathbf{x}-\mathbf{y}|+\delta)^{\frac{1}{2\pi}(2\gamma-\alpha)^2} |\mathbf{x}-\mathbf{y}|^{2-\widetilde{\varepsilon}} \delta^{-\frac{\gamma^2}{\pi}} \frac{\sigma(d\mathbf{x})\sigma(d\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{2-\widetilde{\varepsilon}}} \\
&\le K_2K_3(5\delta)^{\frac{1}{2\pi}(2\gamma-\alpha)^2} (4\delta)^{2-\widetilde{\varepsilon}} \delta^{-\frac{\gamma^2}{\pi}} \le K_4\, \delta^{\frac{1}{2\pi}(2\gamma-\alpha)^2-\frac{\gamma^2}{\pi}+2-\widetilde{\varepsilon}}.
\end{aligned}
$$

Since $2(\varepsilon+\delta) \le |\mathbf{x}-\mathbf{y}| < \eta$ in the integrand of II,

$$
\begin{aligned}
\mathrm{II} &\le K_2 \int\int_{A\times A\cap\{|\mathbf{x}-\mathbf{y}|<\eta\}} (2|\mathbf{x}-\mathbf{y}|)^{(2\gamma-\alpha)^2/2\pi} \left(\frac{|\mathbf{x}-\mathbf{y}|}{2}\right)^{-\gamma^2/\pi} |\mathbf{x}-\mathbf{y}|^{2-\widetilde{\varepsilon}} \frac{\sigma(d\mathbf{x})\sigma(d\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{2-\widetilde{\varepsilon}}} \\
&\le K_5\, \eta^{\frac{1}{2\pi}(2\gamma-\alpha)^2-\frac{\gamma^2}{\pi}+2-\widetilde{\varepsilon}}.
\end{aligned}
$$

Hence (4.37) holds true.

Thus, by virtue of Theorem 4.13, the convergence in probability of random measures $\mu_\varepsilon(\cdot,\omega)$ as $\varepsilon \downarrow 0$ to a non-trivial random measure $\overline{\mu}$ on $B(S-1)$ relative to the metric (4.48) has been verified for $\gamma \in (0, 2\sqrt{\pi})$.

Finally, in order to verify (5.1) and (5.2), we consider the spaces $\overset{\circ}{\mathcal{M}}_0\,(\mathbb{C})$ of measures on $\mathbb{C}$ stated in §2.4 (III). By [PS, Prop.3.4.9],

$$
\nu^{\mathbf{x},r} \in \overset{\circ}{\mathcal{M}}_0\,(\mathbb{C}), \quad \text{and} \quad U\nu^{\mathbf{x},r}(\mathbf{z}) = \frac{1}{\pi}\log\frac{1}{|\mathbf{x}-\mathbf{z}|\vee r}, \quad \mathbf{z}\in\mathbb{C}. \tag{5.9}
$$

Furtherhmore, by virtue of [PS, Prop.3.4.11,Th.3.4.12], we see that, for $B(\mathbf{x},r) \subset B(S)$, $\nu^{\mathbf{x},r}$ admits a probabilistic expression, by using the uniform probability measure $s_{\partial B(t)}$ on the circle $\partial B(t)$,

$$
\nu^{\mathbf{x},r}(A) = \int_{\partial B(t)} \mathbb{P}_{\mathbf{y}}(X_{\sigma_{\partial B(\mathbf{x},r)}} \in A) s_{\partial B(t)}(d\mathbf{y}), \quad A \in \mathcal{B}(\partial B(\mathbf{x},r)), \quad \forall t \ge S,
$$

in terms of the planar Brownian motion $\mathbb{M} = (X_t, \mathbb{P}_\mathbf{x})$. By integrating the both hand sides by tdt from S to $S+1$ and deviding by $\frac{1}{2}(2S+1)$, we obtain the identity (5.1) from the probabilistic expression (4.16) of $\mu^{\mathbf{x},r}$.

By (5.9), we have $Us_{\partial B(t)}(\mathbf{z}) = \frac{1}{\pi}\log\frac{1}{|\mathbf{z}|\vee t}$. Consequently, by the same computation as above, we see that $\widetilde{m}_F \in \overset{\circ}{\mathcal{M}}_0\,(\mathbb{C})$ and $U\widetilde{m}_F(\mathbf{z})$ takes a constant value $\ell = \frac{2}{(2S+1)\pi}\int_S^{S+1} t\log\frac{1}{t}dt$ for $\mathbf{z} \in B(S)$. It then follows from Lemma 3.7 and $\langle \widetilde{m}_F, U\nu^{\mathbf{x},r}\rangle = \langle U\widetilde{m}_F, \nu^{\mathbf{x},r}\rangle = \ell$ that, for $z \in B(S)$,

$$
\widehat{R}\mu^{\mathbf{x},r}(\mathbf{z}) = U\nu^{\mathbf{x},r}(\mathbf{z}) - U\widetilde{m}_F(\mathbf{z}) - \ell + \langle \widetilde{m}_F, U\widetilde{m}_F\rangle = U\nu^{\mathbf{x},r}(\mathbf{z}) - 2\ell + \langle \widetilde{m}_F, U\widetilde{m}_F\rangle,
$$

yielding (5.2). □

Example 5.2 We next examine the case that $E = \overline{\mathbb{H}}$, m is the Lebesgue measure on $\mathbb{H}$ and $(\mathcal{E},\mathcal{F})$ is the regular recurrent Dirichlet form $(\frac{1}{2}\mathbf{D}_\mathbb{H}, H^1(\mathbb{H}))$ on $L^2(\overline{\mathbb{H}}) = L^2(\overline{\mathbb{H}}; m)$, where

$$
\mathbf{D}_\mathbb{H}(u,v) = \int_\mathbb{H} \nabla u(\mathbf{x})\cdot\nabla v(\mathbf{x})d\mathbf{x}, \quad H^1(\mathbb{H}) = \{u \in L^2(\mathbb{H}); |\nabla u| \in L^2(\mathbb{H})\}.
$$

The associated diffusion $\widehat{\mathbb{M}} = (\widehat{X}_t, \widehat{\mathbb{P}}_x)$ is the reflecting Brownian motion (RBM in abbreviation) on $\overline{\mathbb{H}}$.

Clearly the conditions **(B.1)** and **(B.2)** are satisfied. We take $F_+ = \{\mathbf{x} \in \overline{\mathbb{H}} : S < |\mathbf{x}| \le S+1\}$ as an admissible set. Let $\{R\mu : \mu \in \mathcal{M}_0(\overline{\mathbb{H}})\}$ be the family of recurrent potentials relative to F_+ and, for $\mathbf{x} \in E_0 = \{\mathbf{x} \in \mathbb{H} : |\mathbf{x}| < S-1,\ \Im\mathbf{x} > 1\}$ and $r \in (0,1)$, let $\mu^{\mathbf{x},r}$ be the equilibrium measure for $\overline{B(\mathbf{x},r)} (\subset B_+(S) = \{\mathbf{y} \in \mathbb{H},\ |\mathbf{y}| < S\})$ relative to F_+.

For a finite signed measure μ on $\overline{\mathbb{H}}$ with compact support, its logarithmic potential $\widehat{U}\mu$ for RBM is defined by

$$\widehat{U}\mu(\mathbf{x}) = \int_{\overline{\mathbb{H}}} \widehat{k}(\mathbf{x},\mathbf{y})\mu(d\mathbf{y}),\ \mathbf{x} \in \overline{\mathbb{H}},\ \text{for } \widehat{k}(\mathbf{x},\mathbf{y}) = \frac{1}{\pi}\log\frac{1}{|\mathbf{x}-\mathbf{y}|} + \frac{1}{\pi}\log\frac{1}{|\mathbf{x}-\mathbf{y}^*|}, \tag{5.10}$$

where, for $\mathbf{y} = (y_1, y_2)$, $\mathbf{y}^* = (y_1, -y_2)$ denotes its reflection relative to $\partial\mathbb{H}$. The collection of finite signed measures μ on $\overline{\mathbb{H}}$ with compact support and with $\langle |\mu|, \widehat{U}|\mu| \rangle < \infty$ will be denoted by $\overset{\circ}{\mathcal{M}}_0\,(\overline{\mathbb{H}})$.

For each $\mathbf{x} \in E_0$ and $r \in (0,1)$, define a function $\widehat{R}\mu^{\mathbf{x},r}$ on $\overline{\mathbb{H}}$ by (4.17) in terms of $\widehat{\mathbb{M}}$, which is a version of $R\mu^{\mathbf{x},r}$ and continuous on $B_+(S)$. It then holds that for every $\mathbf{z} \in B_+(S)$

$$\widehat{R}\mu^{\mathbf{x},r}(\mathbf{z}) = \frac{1}{\pi}\log\frac{1}{|\mathbf{z}-\mathbf{x}| \vee r} + \frac{1}{\pi}\log\frac{1}{|\mathbf{z}-\mathbf{x}^*|} - \frac{1}{\pi}\widehat{\mathbb{E}}_{\mathbf{z}}\left[\log\frac{1}{|\widehat{X}_\sigma - \mathbf{x}^*|}\right] + C_{\mathbf{x},r}, \tag{5.11}$$

$$\text{where} \qquad C_{\mathbf{x},r} = \langle \mu^{\mathbf{x},r}, \log\frac{1}{|\cdot - \mathbf{x}^*|}\rangle - 4\ell(S) + \langle \widetilde{m}_{F_+}, \widehat{U}\widetilde{m}_{F_+}\rangle. \tag{5.12}$$

Here, σ is the hitting time $\sigma_{\overline{B}(\mathbf{x},r)}$ of $\widehat{\mathbb{M}}$ for $\overline{B}(\mathbf{x},r)$, $\ell(S)$ is the constant defined by (5.3) and $\widetilde{m}_{F_+}(A) = \frac{1}{m(F_+)}m(A \cap F_+),\ A \in \mathcal{B}(\overline{\mathbb{H}})$. A proof will be given in the last part of this example.

(5.11) means that the Robin constant $f(\mathbf{x},r)$ for $B(\mathbf{x},r) \subset B_+(S)$ equals

$$f(\mathbf{x},r) = \frac{1}{\pi}\log\frac{1}{r} + C_{\mathbf{x},r}. \tag{5.13}$$

In view of (5.12), $C_{\mathbf{x},r}$ is uniformly bounded in $(\mathbf{x},r) \in E_0 \times (0,1)$, and consequently the condition **(B.3)** with E_0 in place of $B(S-1)$ is fulfilled by virtue of (5.13). Since the second and third terms on the right hand side of (5.11) are bounded in $\mathbf{x} \in E_0$, $z \in B_+(S)$ and $r \in (0,1)$, (5.11) implies (4.19) with $\kappa = 1$ so that (4.20) with $\kappa = 1$ and with $E_0, B_+(S)$ in place of $B(S-1), B(S)$, respectively, is also fulfilled.

The extended Dirichlet space $\mathcal{F}_e$ is now the Beppo Levi space $\mathrm{BL}(\mathbb{H})$ over $\mathbb{H}$ defined by

$$\mathrm{BL}(\mathbb{H}) = \{u \in L^2_{\mathrm{loc}}(\mathbb{H}) : |\nabla u| \in L^2(\mathbb{H})\}.$$

Let $\{X_u; u \in \mathrm{BL}(\mathbb{H})\}$ be the centered Gaussian field defined on a probability space $(\Omega, \mathcal{B}, \mathbb{P})$ with covariance $\mathbb{E}[X_u X_v] = \frac{1}{2}\mathbf{D}_{\mathbb{H}}(u,v)\ u,v \in \mathrm{BL}(\mathbb{H})$. Define $Y^{\mathbf{x},r}$ by (4.29). As the Robin constant $f(\mathbf{x},r)$ differs from $\frac{1}{\pi}\log\frac{1}{r}$ by $C_{\mathbf{x},r}$ that is uniformly bounded in $(\mathbf{x},r) \in E_0 \times (0,1)$, we can repeat the same argument as in the preceding example to conclude that, for any non-trivial positive finite measure σ on E_0 absolutely continuous with respect to the Lebesgue measure satisfying (5.5) and (5.6) with E_0 in place of $B(S-1)$, the random measures

$$\mu_\varepsilon(A,\omega) = \int_A \exp\left(\gamma Y^{\mathbf{x},\varepsilon} - \frac{\gamma^2}{2}f(\mathbf{x},\varepsilon)\right)\sigma(d\mathbf{x}),\ A \in \mathcal{B}(B(E_0)),$$

on E_0 is convergent in probability to a non-trivial random measure $\overline{\mu}$ on E_0 relative to the present counterpart of the metric (4.48) as $\varepsilon \downarrow 0$ when $\gamma \in (0, 2\sqrt{\pi})$.

Finally, in order to verify (5.11) and (5.12), we first note two facts. It follows from (5.9), (5.10) and [F2, Lem.3.4] that

$$\nu^{\mathbf{x},r} \in \overset{\circ}{\mathcal{M}}_0 (\overline{\mathbb{H}}), \quad \widehat{U}\nu^{\mathbf{x},r}(\mathbf{z}) = \frac{1}{\pi}\log\frac{1}{|\mathbf{z}-\mathbf{x}|\vee r} + \frac{1}{\pi}\log\frac{1}{|\mathbf{z}-\mathbf{x}^*|\vee r}, \quad \mathbf{z}\in B_+(S). \tag{5.14}$$

By [F2, Lem. 3.4] again, $\widehat{U}m_{F_+}(\mathbf{z}) = Um_F(\mathbf{z}),\ \mathbf{z}\in\overline{\mathbb{H}}$ so that

$$\widetilde{m}_{F_+} \in \overset{\circ}{\mathcal{M}}_0 (\overline{\mathbb{H}}), \quad \widehat{U}\widetilde{m}_{F_+}(\mathbf{z}) = 2\ell(S) \quad \text{for every } \mathbf{z}\in B_+(S), \tag{5.15}$$

where $\ell(S)$ is the constant defined by (5.3).

By the RBM version [F2, Prop.3.2] of the fundamental identity for the logarithmic potentials, we have, for any compact set $K\subset\overline{\mathbb{H}}$, $\widehat{U}\mu(\mathbf{z}) = \widehat{H}_K\widehat{U}\mu(\mathbf{z}) + R^{\overline{\mathbb{H}}\setminus K}\mu(\mathbf{z}) - \dot{W}_K(\mathbf{z})\langle\mu,1\rangle$. Substituting $\mu = \nu^{\mathbf{x},r} - \widetilde{m}_{F_+},\ K = \overline{B}(\mathbf{x},r)$ and taking (5.14) and (5.15) into account, we get

$$\begin{aligned}
\widehat{U}\nu^{\mathbf{x},r}(\mathbf{z}) &= \widehat{U}\widetilde{m}_{F_+}(\mathbf{z}) + \frac{1}{\pi}\log\frac{1}{r} + \frac{1}{\pi}\widehat{\mathbb{E}}_{\mathbf{z}}\left[\log\frac{1}{|\widehat{X}_\sigma - \mathbf{x}^*|}\right] - \widehat{U}\widetilde{m}_{F_+}(\mathbf{z}) - R^{\overline{\mathbb{H}}\setminus\overline{B}(\mathbf{x},r)}\widetilde{m}_{F_+}(\mathbf{z}) \\
&= \frac{1}{\pi}\log\frac{1}{r} + \frac{1}{\pi}\widehat{\mathbb{E}}_{\mathbf{z}}\left[\log\frac{1}{|\widehat{X}_\sigma - \mathbf{x}^*|}\right] - R^{\overline{\mathbb{H}}\setminus\overline{B}(\mathbf{x},r)}\widetilde{m}_{F_+}(\mathbf{z}), \quad \mathbf{z}\in B_+(S),
\end{aligned}$$

where $\sigma = \sigma_{\overline{B}(\mathbf{x},r)}$. On the other hand, we have by definition

$$\widehat{R}\mu^{\mathbf{x},r}(\mathbf{z}) = f(\mathbf{x},r) - R^{\overline{\mathbb{H}}\setminus\overline{B}(\mathbf{x},r)}\widetilde{m}_{F_+}(\mathbf{z}), \quad z\in B_+(S).$$

It follows from the above two identities that

$$\widehat{R}\mu^{\mathbf{x},r}(\mathbf{z}) = \widehat{U}\nu^{\mathbf{x},r}(\mathbf{z}) + f(\mathbf{x},r) - \frac{1}{\pi}\log\frac{1}{r} - \frac{1}{\pi}\widehat{\mathbb{E}}_{\mathbf{z}}\left[\log\frac{1}{|\widehat{X}_\sigma - \mathbf{x}^*|}\right], \quad z\in B_+(S). \tag{5.16}$$

We next show that the uniform probability measure $\nu^{\mathbf{x},r}$ on $\partial B(\mathbf{x},r)$ belongs to the space $\mathcal{M}_0(\overline{\mathbb{H}})$ when $\overline{B(\mathbf{x},r)}\subset B_+(S)$. We know from (5.1) that $\nu^{\mathbf{x},r}\in\mathcal{M}_0(\mathbb{C})$, or equivalently, there eixists a constant $C>0$ with

$$\left(\int|\varphi|d\nu^{\mathbf{x},r}\right)^2 \le C\left(\frac{1}{2}\mathbf{D}_{\mathbb{C}}(\varphi,\varphi) + \int\varphi^2 dm_F\right), \quad \text{for any } \varphi\in C_c^1(\mathbb{C}). \tag{5.17}$$

Define $\mathcal{C}(\overline{\mathbb{H}}) = C_c^1(\mathbb{C})\big|_{\overline{\mathbb{H}}}$ and extend any $\varphi\in\mathcal{C}(\overline{\mathbb{H}})$ to $\widehat{\varphi}\in C_c^1(\mathbb{C})$ by reflection relative to $\partial\mathbb{H}$: $\widehat{\varphi}(\mathbf{y}) = \varphi(\mathbf{y}^*)$, $\Im\mathbf{y}<0$. We then get from (5.17) the same inequality holding for $\varphi\in\mathcal{C}(\overline{\mathbb{H}})$ with $2C, \mathbf{D}_{\mathbb{H}}, m_{F_+}$ in place of $C, \mathbf{D}_{\mathbb{C}}, m_F$, respectively, which means that $\nu^{\mathbf{x},r}\in\mathcal{M}_0(\overline{\mathbb{H}})$ on account of [FOT, Lem.6.1.1].

Lemma 3.7 applied to $\nu^{\mathbf{x},r}\in\mathcal{M}_0(\overline{\mathbb{H}})\cap\overset{\circ}{\mathcal{M}}_0(\overline{\mathbb{H}})$ along with (5.15) leads us to

$$R\nu^{\mathbf{x},r}(\mathbf{z}) = \widehat{U}\nu^{\mathbf{x},r}(\mathbf{z}) + \widehat{C}, \text{ with } \widehat{C} = -4\ell(S) + \langle\widetilde{m}_{F_*}.\widehat{U}\widetilde{m}_{F_+}\rangle,$$

for q.e. $\mathbf{z}\in\overline{\mathbb{H}}$. This combined with (5.14) yields

$$f(\mathbf{x},r) = \langle R\mu^{\mathbf{x},r},\nu^{\mathbf{x},r}\rangle = \langle\mu^{\mathbf{x},r}, R\nu^{\mathbf{x},r}\rangle = \langle\mu^{\mathbf{x},r},\widehat{U}\nu^{\mathbf{x},r}\rangle + \widehat{C} = \frac{1}{\pi}\log\frac{1}{r} + C_{\mathbf{x},r}.$$

By substituting this into (5.16), we arrive at (5.11) and (5.12).

Example 5.3 Finally in this subsection, we consider the case that $E = \mathbb{C}$, m is the Lebesgue measure on $\mathbb{C}$ and $(\mathcal{E}, \mathcal{F})$ is the regular recurrent Dirichlet form $(\mathbf{a}, H^1(\mathbb{C}))$ on $L^2(\mathbb{C}) = L^2(\mathbb{C}; m)$, where the form $\mathbf{a}$ is defined by (1.2) by means of measurable coefficients $a_{ij}(\mathbf{x})$, $1 \le i, j \le 2$, on $\mathbb{C}$ satisfying the uniform ellipticity (1.3) for some positive constants $\lambda \le \Lambda$. The associated diffusion on $\mathbb{C}$ will be denoted by $\mathbb{M} = (X_t, \mathbb{P}_x)$ (cf. [FOT, Exa.4.5.2]).

It is well known that the condition **(B.1)** is fulfilled by this example (see [BGK] and references therein). According to [LSW], the regularity of the boundary point for the Dirichlet problem on an open set is equivalent to that for the case of Example 5.1. Therefore condition **(B.2)** is also fulfilled by Proposition 4.3.

Analogously to Example 5.1, we take $F = \overline{B(S+1)} \setminus B(S)$ as an admissible set and let $\{R\mu : \mu \in \mathcal{M}_0(\mathbb{C})\}$ be the family of recurrent potentials relative to F, $\mu^{\mathbf{x},r}$ be the equilibrium measure for $\overline{B(\mathbf{x},r)} (\subset B(S))$ relative to F and $f(\mathbf{x}, r)$ be the corresponding Robin constant.

Example 5.1 is a special case of the present one where $a_{ij}(\mathbf{x}) = \frac{1}{2}\delta_{ij}$. According to (5.4), the Robin constant for $B(\mathbf{x}, r)$ in this special case is given by $\frac{1}{\pi} \log \frac{1}{r} + \ell_1(S)$ for a constant $\ell_1(S) = -2\ell(S) + \langle \widetilde{m}_F, U\widetilde{m}_F \rangle$ with $\ell(S)$ of (5.3). Therefore we obtain from Proposition 3.6 the bound

$$\frac{1}{2\Lambda}\left(\frac{1}{\pi}\log\frac{1}{r} + \ell_1(S)\right) \le f(\mathbf{x}, r) \le \frac{1}{2\lambda}\left(\frac{1}{\pi}\log\frac{1}{r} + \ell_1(S)\right). \tag{5.18}$$

which particularly means that the condition **(B.3)** is fulfilled with $C_2 = \Lambda/\lambda$.

We have verified by Lemma 4.6 (iii) that (4.19) is valid for $\kappa = C_H^7$ in a general setting so that (4.20) is fulfilled with this big constant κ accordong to Lemma 4.5 (ii). However, if we make some smoothness assumption on the coefficients $a_{ij}(\mathbf{x})$, $1 \le i, j \le 2$ in the present case, we can attain a much better choice of κ: $\kappa = \Lambda/\lambda$ which equals 1 in the case that $a_{ij}(\mathbf{x}) = C\delta_{ij}$ for a constant $C > 0$ as in Example 5.1.

Proposition 5.4 *If $a_{ij}(\mathbf{x})$ and their first derivatives are Hölder continuous on $\mathbb{C}$, then* (4.19) *holds for $\kappa = \Lambda/\lambda$ so that* (4.20) *is filfilled with this constant κ.*

A proof of this proposition will be given in Appendix (subsection 7.2) by making use of a construction of a fundamental solution for $\mathbf{a}$ from a parametrix

$$\Gamma_0(\mathbf{x}, \mathbf{y}) = -\frac{1}{(4\pi \mathrm{det}(\mathrm{a}^{\mathrm{ij}}(\mathbf{y})))^{1/2}} \log\left(\sum_{i,j=1}^{2} a^{ij}(\mathbf{y})(x_i - y_i)(x_j - y_j)\right), \tag{5.19}$$

as is stated in [Fr, §5.6]. Here $(a^{ij}(\mathbf{y}))$ denotes the inverse matrix of $(a_{ij}(\mathbf{y}))$.

The conditions (4.36) and (4.37) (for the constant κ in (4.20)) can be verified to hold for certain values of $\gamma > 0$ as Example 5.1 by using (5.18). In fact, if $0 < \gamma < \alpha < 2\gamma$ and σ satisfies (5.5) and (5.6), then

$$\begin{aligned}
&\int_A \exp\left(-\frac{1}{2}(2\gamma - \alpha)^2 f(\mathbf{y}, 7\delta) + \gamma^2 f(\mathbf{y}, \delta)\right) \sigma(B(\mathbf{y}, 6\delta))\sigma(d\mathbf{y}) \\
&\le K_6 \exp\left(-\frac{1}{4\pi\Lambda}(2\gamma - \alpha)^2 \log\frac{1}{7\delta} + \frac{\gamma^2}{2\pi\lambda}\log\frac{1}{\delta}\right)\delta^2 \\
&\le K_7 \delta^{(2\gamma-\alpha)^2/4\pi\Lambda - \gamma^2/2\pi\lambda + 2}.
\end{aligned} \tag{5.20}$$

Similarly, the integral in (4.37) is estimated as

$$
\begin{aligned}
&\int_{A\times A\cap\{|\mathbf{x}-\mathbf{y}|<\eta\}} \exp\left(-\frac{1}{2}(2\gamma-\alpha)^2 f(\mathbf{y},|\mathbf{x}-\mathbf{y}|+\delta)\right) \\
&\qquad \times \exp\left(\gamma^2\kappa f(\mathbf{y},(|\mathbf{x}-\mathbf{y}|-(\varepsilon+\delta))\vee\delta)\right)\sigma(d\mathbf{x})\sigma(d\mathbf{y}) \\
&\le K_8 \int_{A\times A\cap\{|\mathbf{x}-\mathbf{y}|<2(\varepsilon+\delta)\}} \exp\left(\frac{1}{4\pi\Lambda}(2\gamma-\alpha)^2\log(|\mathbf{x}-\mathbf{y}|+\delta)-\frac{\gamma^2\kappa}{2\pi\lambda}\log\delta\right)\sigma(d\mathbf{x})\sigma(d\mathbf{y}) \\
&+K_8 \int_{A\times A\cap\{2(\varepsilon+\delta)\le|\mathbf{x}-\mathbf{y}|<\eta\}} \exp\left(\frac{1}{4\pi\Lambda}(2\gamma-\alpha)^2\log(|\mathbf{x}-\mathbf{y}|+\delta)-\frac{\gamma^2\kappa}{2\pi\lambda}\log(|\mathbf{x}-\mathbf{y}|-(\varepsilon+\delta))\right)\sigma(d\mathbf{x})\sigma(d\mathbf{y}) \\
&= \mathrm{I}'+\mathrm{II}'.
\end{aligned}
$$

In the same way as in Example 5.1, we see that, for $\widetilde{\varepsilon}\in(0,2)$,

$$
\mathrm{I}'\le K_8\exp\left(\frac{(2\gamma-\alpha)^2}{4\pi\Lambda}\log(5\delta)-\frac{\gamma^2\kappa}{2\pi\lambda}\log\delta\right)(4\delta)^{2-\widetilde{\varepsilon}}\le K_9\delta^{(2\gamma-\alpha)^2/4\pi\Lambda-\gamma^2\kappa/2\pi\lambda+2-\widetilde{\varepsilon}}.
$$

Furthermore, we can see that

$$
\mathrm{II}'\le K_9\eta^{(2\gamma-\alpha)^2/4\pi\Lambda-\gamma^2\kappa/2\pi\lambda+2-\widetilde{\varepsilon}},
$$

provided that the exponent appearing on the righthand side is positive for some $\widetilde{\varepsilon}>0$.

Therefore (4.36) and (4.37) are satisfied if the inequality $\frac{1}{4\pi\Lambda}(2\gamma-\alpha)^2-\frac{\kappa\gamma^2}{2\pi\lambda}+2>0$ holds for some $\alpha\in(\gamma,2\gamma)$ with $\kappa\ge 1$. This condition is fulfilled provided that

$$
\gamma\in\left(0,\ 2\sqrt{\frac{2\pi\lambda\Lambda}{2\kappa\Lambda-\lambda}}\right). \tag{5.21}
$$

Let $\{X_u; u\in\mathrm{BL}(\mathbb{C})\}$ be the centered Gaussian field defined on a probability space $(\Omega,\mathcal{B},\mathbb{P})$ with covariance $\mathbb{E}[X_uX_v]=\mathbf{a}(u,v)\ u,v\in\mathrm{BL}(\mathbb{C})$. Define $Y^{\mathbf{x},r}$ by (4.29). Define $\mu_\varepsilon(A,\omega)=\int_A\exp(\gamma Y^{\mathbf{x},\varepsilon}-\frac{\gamma^2}{2}f(\mathbf{x},\varepsilon))\sigma(d\mathbf{x})$ for a non-trivial positive finite measure σ on $B(S-1)$ satisfying (5.5) and (5.6) and for $A\in\mathcal{B}(B(S-1))$, $\omega\in\Omega$ by taking Proposition 4.9 into account. As in Example 5.1, the convergence in probability of the random measures $\mu_\varepsilon(\cdot,\omega)$ toward a non-trivial random measure $\overline{\mu}$ relative to the metric (4.48) as $\varepsilon\downarrow 0$ is legitimate for γ in this region according to Theorem 4.13. If $a_{ij}\in C^2(\mathbb{C})$, then, by Propostion 5.4, the range (5.21) equals $\left(0,2\sqrt{\frac{2\pi\lambda^2\Lambda}{2\Lambda^2-\lambda^2}}\right)$, which reduces to $(0,2\sqrt{\pi})$ when $a_{ij}(\mathbf{x})=\frac{1}{2}\delta_{ij}$.

6 GMCs via equilibrium potentials for transient forms

6.1 Construction of Gaussian multiplicative chaos for transient forms

In this subtsection, we assume that E is an open subset of $\mathbb{C}$ and m is the Lebesgue measure on E. We consider a regular transient strongly local Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(E;m)$ and the associated diffusion process $\mathbb{M}=(X_t,\mathbb{P}_x)$ on E. We fix a bounded open set E_0 with $\overline{E}_0\subset E$ and aim at constructing Gaussian multiplicative chas on E_0. We choose $a>0$ such that the a-neighborhood of E_0 is contained in E.

The transition function $\{P_t,\ t>0\}$ of $\mathbb{M}$ is assumed to satisfy the absolute continuity condition **(AC)** with $N=\emptyset$ and some more:

(C.1) **(i)** $P_t(\mathbf{x},\cdot)$ is absolutely continuous with respect to m for each $t>0$ and $\mathbf{x}\in E$ with a density function $p_t(\mathbf{x},\mathbf{y})$ jointly continuous in $t>0,\ \mathbf{x},\mathbf{y}\in E$.

(ii) $p_t(\mathbf{x},\mathbf{y})$ admits a Gaussian upper bound: for some constants $K>0,\ k>0$,

$$p_t(\mathbf{x},\mathbf{y})\le \frac{K}{t}e^{-k|\mathbf{x}-\mathbf{y}|^2/t},\quad \mathbf{x},\mathbf{y}\in E, t>0.$$

The resolvent kernel $\{R_\alpha,\alpha>0\}$ of $\mathbb{M}$ then admits a density function $r_\alpha(\mathbf{x},\mathbf{y}),\ \mathbf{x},\mathbf{y}\in E$, with respect to m possessing properties [FO, (2.10),(2.11)]. We let $r(\mathbf{x},\mathbf{y})=\lim_{\alpha\downarrow 0}r_\alpha(\mathbf{x},\mathbf{y})$ and write $R\mu(\mathbf{x})=\int r(\mathbf{x},\mathbf{y})\mu(d\mathbf{y}),\ \mathbf{x}\in E$, for a positive Radon measure μ on E. Let $\mathcal{S}_0^{(0)}$ be the family of positive Radon measures on E of finite 0-order energy and $U\mu\in\mathcal{F}_e$ be the 0-order potential of $\mu\in\mathcal{S}_0^{(0)}$. As the proof of [FO, Prop.2,5 (ii)], we see that $\mu\in\mathcal{S}_0^{(0)}$ if and only if $\langle\mu,R\mu\rangle<\infty$ and in this case $R\mu$ is the excessive version of $U\mu$.

Consider a compact set $B\subset E$ whose 0-order capacity $\mathrm{Cap}^{(0)}(B)$ is positive. As has been explained in the last part of Section 3, there exists a unique measure $\mu_B\in\mathcal{S}_0^{(0)}$ supported by B such that

$$R\mu_B(\mathbf{y})=\mathbb{P}_\mathbf{y}(\sigma_B<\infty),\quad \text{for every } \mathbf{y}\in E. \tag{6.1}$$

The equality holds for every $\mathbf{y}\in E$ because the both hand sides are excessive functions of $\mathbf{y}$. μ_B has been called the 0-order equilibrium measure of B, but we consider instead the *renormalized equlibrium measure* μ^B of B defined by (3.20). μ^B is a probability measure concentrated on B (actually on ∂B) and

$$R\mu^B(\mathbf{y})=\frac{1}{\mathrm{Cap}^{(0)}(B)}\mathbb{P}_\mathbf{y}(\sigma_B<\infty),\quad \text{for every } \mathbf{y}\in E,$$

so that present counterpart of the *Robin constant* of B equals $1/\mathrm{Cap}^{(0)}(B)$.

Let us denote the closed disk $\{y\in\mathbb{C}:|y-x|\le r\}$ by $\overline{B}(\mathbf{x},r)$. For $\overline{B}(\mathbf{x},r)\subset E$, define

$$\mu^{\mathbf{x},r}=\mu^{\overline{B}(\mathbf{x},r)},\quad f(\mathbf{x},r)=1/\mathrm{Cap}^{(0)}(\overline{B}(\mathbf{x},r)). \tag{6.2}$$

We then have

$$R\mu^{\mathbf{x},r}(\mathbf{y})=f(\mathbf{x},r)\mathbb{P}_\mathbf{y}(\sigma_{\overline{B}(\mathbf{x},r)}<\infty),\quad \text{for every } \mathbf{y}\in E. \tag{6.3}$$

We now make the following additional assumptions.

(C.2) Any non-negative $\mathcal{E}$-harmonic function $u\in\mathcal{F}_e$ on an open set $G\subset E$ has an m-version $\widetilde{u}$ that is continuous on G and satisfies the Harnack inequality (4.2) for any $B(\mathbf{x},r)\subset G$.

(C.3) For any $\mathbf{x}\in E_0$, the one-point set $\{\mathbf{x}\}$ is of zero capacity relative to $\mathcal{E}$.

(C.4) For any disk B with $\overline{B}\subset E$, every point of ∂B is regular for $\overline{B}$ and for the Dirichlet problem on $E\setminus\overline{B}$.

(C.5) There exists a constant $C_2>0$ such that

$$\sup\{f(\mathbf{y},r):\mathbf{y}\in E_0\}\le C_2\inf\{f(\mathbf{y},r):\mathbf{y}\in E_0\},\quad r\in(0,a).$$

We state important properties of $\{\mu^{\mathbf{x},r},f(\mathbf{x},r)\}$ defined by (6.2) under the above assumptions.

Proposition 6.1 (i) $\lim_{r\downarrow 0} f(\mathbf{x}, r) = \infty$ *for any* $\mathbf{x} \in E$.
(ii) *For any* $r_0 \in (0, a/3)$,*the uniform bound* (4.22) *holds true for* R *and* E_0 *in place of* R^g *and* $B(S-1)$, *respectively.*
(iii) *There exist constants* $\kappa \geq 1$ *and* $C_1 > 0$ *such that, for all* $\mathbf{x}, \mathbf{y} \in E_0$ *and* $0 < \varepsilon \leq \delta$ *with* $4\delta \leq |\mathbf{x} - \mathbf{y}| < a/3$,

$$\langle \mu^{\mathbf{x},\varepsilon}, R\mu^{\mathbf{y},\delta} \rangle \leq \kappa f(\mathbf{y}, |\mathbf{x} - \mathbf{y}| - (\varepsilon + \delta)) + C_1. \tag{6.4}$$

(iv) *For any* $\eta \in (0, a/2)$, Proposition 4.7 (ii) *holds true for* E_0 *and* $r(\mathbf{x}, \mathbf{y})$ *in place of* $B(S-1)$ *and* $\mathfrak{r}(\mathbf{x}, \mathbf{y})$, *respectively.*
(v) *For each* $\mathbf{x} \in E$, $f(\mathbf{x}, r)$ *is strictly decreasing and continuous in* $r > 0$.
(vi) *The mapping* $\mathbf{x} \in E_0 \mapsto R\mu^{\mathbf{x},r} \in \mathcal{F}_e$ *is continuous.*

Proof. (i) follows directly from (6.2) and the assumption **(C.3)**.
(ii). As in the proof of Lemma 4.6, the assumption **(C.2)** implies that, for $\mathbf{y} \in E_0$ and $0 < 2r < r_0 < a/3$,

$$\max\{R\mu^{\mathbf{y},r}(\mathbf{w}) : \mathbf{w} \in \partial B(\mathbf{y}, r_0)\} \leq C_H^7 \min\{R\mu^{\mathbf{y},r}(\mathbf{w}) : \mathbf{w} \in \partial B(\mathbf{y}, r_0)\}. \tag{6.5}$$

By a similar argument made below (4.25), we have $R\mu^{\mathbf{y},r}(\mathbf{x}) = H_{\partial B(\mathbf{y},r_0)} R\mu^{\mathbf{y},r}(\mathbf{x})$ for any $\mathbf{x} \in E \setminus B(\mathbf{y}, r_0)$, so that $R\mu^{\mathbf{y},r}(\mathbf{x}) \leq \max_{\mathbf{w}\in\partial B(\mathbf{y},r_0)} R\mu^{\mathbf{y},r}(\mathbf{w}) \leq C_H^7 \langle R\mu^{\mathbf{y},r}, \mu^{\mathbf{y},r_0} \rangle = C_H^7 f(\mathbf{y}, r_0)$, which is bounded in $\mathbf{y} \in E_0$ by **(C.5)**.
(iii). (6.4) for $\kappa = C_H^7$, $C_1 = 0$ follows from (6.5) as Lemma 4.5 (ii) using the similar argument to the above.
(iv) can be proved as in subsection 7.1 using **(C.1)** and (ii).
(v). For $C \in \mathcal{B}(E)$, denote by $p_C(\mathbf{y})$ the the hitting probability $\mathbb{P}_{\mathbf{y}}(\sigma_C < \infty)$, $\mathbf{y} \in E$. In particular, we consider the function $v = p_{\overline{B}(\mathbf{x},r)}$ for $B(\mathbf{x}, r) \subset E$. Then $v \in \mathcal{F}_e$ and $\mathcal{E}(v.v) = \mathrm{Cap}^{(0)}(\overline{B}(\mathbf{x}, r))$. In the same way as the proof of Lemma 4.4, we can deduce from the assumption **(C.2)** that v is $\mathcal{E}$-harmonic and continuous on $E \setminus \overline{B}(\mathbf{x}, r)$. Moreover, the assumption **(C.4)** implies that v is equal to 1 on $\overline{B}(\mathbf{x}, r)$ and continuous on E. But v is not identically 1 on E. If v were identically equals 1 on E, we see, by choosing $v_n \in \mathcal{F}_e \cap C_c(E)$ that is $\mathcal{E}$-convergent to 1, $\mathcal{E}(1, 1) = \lim_{n\to\infty} \mathcal{E}(v_n, 1)$ which vanishes by the strong locality, contradicting to a transience criterion [FOT, (1.5.8)].

In view of (6.2), it suffices to show that $\mathrm{Cap}^{(0)}(\overline{B}(\mathbf{x}, r))$ is strictly increasing and continuous in r. Obviously it is non-decreasing. If $\mathrm{Cap}^{(0)}(\overline{B}(\mathbf{x}, r_1)) = \mathrm{Cap}^{(0)}(\overline{B}(\mathbf{x}, r_2))$ for some $0 < r_1 < r_2$, then, by the 0-order version of [FOT, Th.2.1.5], $p_{\overline{B}(\mathbf{x},r_1)}$ equals $p_{\overline{B}(\mathbf{x},r_2)}$ identically, contradicting to the maximum principle for the non-constant harmonic function $p_{\overline{B}(\mathbf{x},r_1)}$ on $E \setminus \overline{B}(\mathbf{x}, r_1)$. Its right continuity follows from [FOT, §A.1 (c)]. If $r_n \uparrow r$, then $\mathrm{Cap}^{(0)}(\overline{B}(\mathbf{x}, r_n)) \uparrow \mathrm{Cap}^{(0)}(B(\mathbf{x}, r))$. Since $\sigma_{B(\mathbf{x},r)} = \sigma_{\overline{B}(\mathbf{x},r)}$ a.s. by the assumption **(C.4)**, we have from [FOT, Th.4.3.3], $\mathrm{Cap}^{(0)}(B(\mathbf{x}, r)) = \mathcal{E}(p_{B(\mathbf{x},r)}, p_{B(\mathbf{x},r)}) = \mathcal{E}(p_{\overline{B}(\mathbf{x},r)}, p_{\overline{B}(\mathbf{x},r)}) = \mathrm{Cap}^{(0)}(\overline{B}(\mathbf{x}, r))$, yielding the left continuity.
(vi), It follows from (6.2) and (6.3) that

$$\mathcal{E}(R\mu^{\mathbf{y},r} - R\mu^{\mathbf{x},r}, R\mu^{\mathbf{y},s} - R\mu^{\mathbf{x},r}) = f(\mathbf{y}, r) + f(\mathbf{x}, r) - 2\langle \mu^{\mathbf{x},r}, R\mu^{\mathbf{y},r} \rangle.$$

Denote $p_{\bar{B}(\mathbf{x},r)}$ and $\mu_{\bar{B}(\mathbf{x},r)}$ by $p_{\mathbf{x},r}$ and $\mu_{\mathbf{x},r}$, repectively. Define

$$e_{x,r}(\mathbf{z}) = \mathbb{E}_{\mathbf{z}}[e^{-\sigma_{\bar{B}(\mathbf{x},r)}}], \quad R^{E\setminus\bar{B}(\mathbf{x},r)} f(z) = \mathbb{E}_{\mathbf{z}}[\int_0^{\sigma_{\bar{B}(\mathbf{x},r)}} f(X_s) ds].$$

Then we have the identity $p_{\mathbf{x},r} = e_{\mathbf{x},r} + R^{E\setminus\bar{B}(\mathbf{x},r)} e_{\mathbf{x},r}$, because the bothhand sides are equal to 1 on $\bar{B}(\mathbf{x}, r)$ on account **(C.4)** and $\mathcal{E}$-harmonic on $E \setminus \bar{B}(\mathbf{x}, r)$ (see [FOT, §2,§4]).

On the other hand, we have from (4.33), $\mathbb{P}_{\mathbf{z}}(\lim_{n\to\infty}\sigma_{\bar{B}(\mathbf{y}_n,r)}=\sigma_{\bar{B}(\mathbf{x},r)})=1,\ \mathbf{z}\in E$, for any sequence $\mathbf{y}_n\in E_0$ converging to $\mathbf{x}\in E_0$. The above identity then implies $\lim_{n\to\infty}p_{\mathbf{y}_n.r}(\mathbf{z})=p_{\mathbf{x},r}(\mathbf{z}),\ \mathbf{z}\in E$, and consequently,

$$\lim_{\mathbf{y}\to\mathbf{x}}p_{\mathbf{y},r}(\mathbf{z})=p_{\mathbf{x},r}(\mathbf{z}),\quad \text{for every } \mathbf{z}\in E. \tag{6.6}$$

Now take a disk $B\subset E$ containing $\bar{B}(\mathbf{x},r)\cup\bar{B}(\mathbf{y},r)$. It follows from (6.6) that, as $\mathbf{y}\to\mathbf{x}$, $\mathrm{Cap}^{(0)}(B(\mathbf{y},r))=\mu_{\mathbf{y},r}(\bar{B}(\mathbf{y},r))=\langle\mu_{\mathbf{y},r},R\mu_{\bar{B}}\rangle=\langle p_{\mathbf{y},r},\mu_{\bar{B}}\rangle$ tends to $\mathrm{Cap}^{(0)}(B(\mathbf{x},r))=\mu_{\mathbf{x},r}(\bar{B}(\mathbf{x},r))=\langle\mu_{\mathbf{x},r},R\mu_{\bar{B}}\rangle=\langle p_{\mathbf{x},r},\mu_{\bar{B}}\rangle$. Namely, $f(\mathbf{y},r)\to f(\mathbf{x},r)$ as $\mathbf{y}\to\mathbf{x}$. By taking $B=B(\mathbf{x},r)$, we also have $\langle\mu_{\mathbf{y},r},R\mu_{\mathbf{x},r}\rangle\to\langle\mu_{\mathbf{x},r},R\mu_{\mathbf{x},r}\rangle,\ \mathbf{y}\to\mathbf{x}$, arriving at the desired continuity (vi). □

Thus all the assertions made in subsection 4.2 for recurrent cases can be carried over to the present transient cases straightforwardly. To be more precise, for the centered Gaussian field $\{X_u: u\in\mathcal{F}_e\}$ with covariance $\mathbb{E}[X_uX_v]=\mathcal{E}(u,v),\ u,v\in\mathcal{F}_e$, define

$$Y^{\mathbf{x},r}=X_{R\mu^{\mathbf{x},\varepsilon}},\quad \widetilde{Y}^{\mathbf{x},\varepsilon,\gamma}=\gamma Y^{\mathbf{x},\varepsilon}-\frac{\gamma^2}{2}f(\mathbf{x},\varepsilon),\quad \mathbf{x}\in E_0,\ \varepsilon\in(0,1),\ \gamma>0. \tag{6.7}$$

by using present transient equilibrium potential $R\mu^{\mathbf{x},r}$ and the Robin constant $f(\mathbf{x},r)$ in (6.2) and (6.3). Note that $\mathbb{E}[(Y^{\mathbf{x},\varepsilon})^2]=f(\mathbf{x},\varepsilon)$.

For a positive Radon measure σ on E_0 absolutely continuous with respect to the Lebesgue measure with a strictly positive bounded density, put

$$\mu_\varepsilon(A,\omega)=\int_A e^{\widetilde{Y}^{\mathbf{x},\varepsilon,\gamma}}\sigma(d\mathbf{x}),\quad A\in\mathcal{B}(E_0). \tag{6.8}$$

Owing to Proposition 6.1 (vi), for each ε, one can choose as Proposition 4.9 a measurable function $Y(\mathbf{x},\varepsilon,\omega)$ on $(\mathbf{x},\omega)\in E_0\times\Omega$ such that, for σ-a.e. $\mathbf{x}\in E_0$, $Y(\mathbf{x},\varepsilon,\omega)=Y^{\mathbf{x},\varepsilon}(\omega)$ $\mathbb{P}$-a.e. The integral in the above makes sense for this version and gives a random measure on E_0.

Analogously to Theorem 4.13, we obtain

Theorem 6.2 *Assume that, for some $\alpha\in(\gamma,2\gamma)$, conditions* (4.36) *and* (4.37) *with the constant κ in* (6.4) *are fulfilled. Then, as $\varepsilon\downarrow 0$, $\mu_\varepsilon(\cdot,\omega)$ converges in probability to a non-degenerate random measure $\overline{\mu}(\cdot,\omega)$ on $(E_0,\mathcal{B}(E_0))$ relative to the metric ρ defined by* (4.48) *for E_0 in place of $D=B(S-1)$.*

In the rest of this subsection and in the next subsection as well, we shall work with transient Dirichlet forms associated with absorbing Brownian motions on planar domains. Let $\mathbb{M}=(X_t,\mathbb{P}_{\mathbf{x}})$ be the Brownian motion on the complex plane $\mathbb{C}$. Let D be a domain in $\mathbb{C}$ with $\mathbb{C}\setminus D$ being non-polar and $\mathbb{M}^D$ be the absorbing Brownian motion on D obtained from $\mathbb{M}$ by killing upon its leaving time τ_D from D. Then $\mathbb{M}^D$ is transient and $(\mathcal{E},\mathcal{F})=(\frac{1}{2}\mathbf{D}_D,H_0^1(D))$ is the regular transient strongly regular Dirichlet form on $L^2(D)$ of $\mathbb{M}^D$. Here

$$\mathbf{D}_D(u,v)=\int_D\nabla u(\mathbf{x})\cdot\nabla v(\mathbf{x})d\mathbf{x},\quad H^1(D)=\{u\in L^2(D):|\nabla u|\in L^2(D)\},$$

and $H_0^1(D)$ is the closure of $C_c^1(D)$ in this space. The extended Dirichlet space $\mathcal{F}_e$ of $(\mathcal{E},\mathcal{F})$ equals the extended Sobolev space $H_{0,e}^1(D)$ that can be obtained by completing the space $C_c^1(D)$ with respect to the 0-order Dirichlet norm $\sqrt{\mathbf{D}_D(u,u)}$. In the following example, we apply Theorem 6.2 to the case that D is a bounded domain of $\mathbb{C}$.

Example 6.3 Assuming that D is a bounded domain of $\mathbb{C}$, we consider the absorbing Brownian motion $\mathbb{M}_D$ on D and the associated Dirichlet form $(\frac{1}{2}\mathbf{D}_D.H^1_0(D))$ on $L^2(D)$ as above. Due to a Poincaré inequality ([FOT, Example 1.5.1], the extended Dirichlet space $H^1_{0,e}(D)$ is the space $H^1_0(D)$ itself which is a real Hilbert space with the 0-order inner product $\frac{1}{2}\mathbf{D}_D(u,v)$.

Using the planar BM $\mathbb{M} = (X_t, \mathbb{P}_{\mathbf{x}})$, the resolvent density $\{r_\alpha(\mathbf{x},\mathbf{y}),\ \alpha > 0,\ \mathbf{x},\mathbf{y} \in D\}$ of $\mathbb{M}^D$ is defined by

$$r_\alpha(\mathbf{x},\mathbf{y}) = g_\alpha(\mathbf{x},\mathbf{y}) - \mathbb{E}_{\mathbf{x}}[e^{-\alpha\tau_D} g_\alpha(X_{\tau_D},\mathbf{y})], \quad g_\alpha(\mathbf{x},\mathbf{y}) = \int_0^\infty e^{-\alpha t}\frac{1}{2\pi t}e^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{2t}}dt.$$

The 0-order resolvent density of $\mathbb{M}^D$ is defined by $r(\mathbf{x},\mathbf{y}) = \lim_{\alpha\downarrow 0} r_\alpha(\mathbf{x},\mathbf{y}), \quad \mathbf{x},\mathbf{y} \in D$.
The *fundamental identity for logarithmic potentials* due to S.C. Port and C.J. Stone [PS, Th.3.4.2] says that

$$r(\mathbf{x},\mathbf{y}) = k(\mathbf{x},\mathbf{y}) - E_{\mathbf{x}}[k(X_{\tau_D},\mathbf{y})], \quad \mathbf{x},\mathbf{y} \in \mathbb{C}, \tag{6.9}$$

where k is the logarithmic kernel defiend by $k(\mathbf{x},\mathbf{y}) = \frac{1}{\pi}\log\frac{1}{|\mathbf{x}-\mathbf{y}|}$, $\mathbf{x},\mathbf{y} \in \mathbb{C}$. Generally the righthand side involves an extra additional term $W_{\mathbb{C}\setminus D}(\mathbf{x})$ which disappears under the present assumption on the boundedness of D.

The present Dirichlet form $(\frac{1}{2}\mathbf{D}_D, H^1_0(D))$ on $L^2(D)$ and the associated diffusion $\mathbb{M}^D$ obviously satisfy the conditions **(C.1) (ii), (C.2), (C.3)** and **(C.4)**. The condition **(C.1) (i)** follows from Theorem 3.1 in [BGK] by taking $X = \mathbb{C}, (\mathcal{E},\mathcal{F}) = (\frac{1}{2}\mathbf{D}_D, H^1(\mathbb{C}))$ and $\Omega = D$. We let $D_0 = \{\mathbf{x} \in D : \mathrm{dist}(\mathbf{x},\partial D) > 1\}$, and consider for the closed disk $\overline{B}(\mathbf{x},r)$ with $\mathbf{x} \in D_0,\ r \in (0,1)$ its renormalized equilibrium measure $\mu^{\mathbf{x},r}$ and the Robin constant $f(\mathbf{x},r)$ defined by (6.2). Then, by (6.3), the renormalized equilibrium potential $R\mu^{\mathbf{x},r} \in H^1_0(D)$ has the expression $R\mu^{\mathbf{x},r}(\mathbf{y}) = f(\mathbf{x},r)\mathbb{P}_{\mathbf{y}}(\sigma_{\overline{B}(\mathbf{x},r)} < \tau_D)$, $\forall \mathbf{y} \in D$, in terms of the planar Brownian motion $\mathbb{M} = (X_t, \mathbb{P}_{\mathbf{x}})$.

We consider again the uniform probability measure $\nu^{\mathbf{x},r}$ on $\partial B(\mathbf{x},r)$. It follows from (6.9) and (5.9) that

$$\nu^{\mathbf{x},r} \in \mathcal{S}^{(0)}, \quad R\nu^{\mathbf{x},r}(\mathbf{z}) = \frac{1}{\pi}\log\frac{1}{|\mathbf{x}-\mathbf{z}|\vee r} - \frac{1}{\pi}\mathbb{E}_{\mathbf{z}}\left[\log\frac{1}{|X_{\tau_D}-\mathbf{x}|}\right],$$

and so

$$\langle \mu^{\mathbf{x},r}, R\nu^{\mathbf{x},r}\rangle = \frac{1}{\pi}\log\frac{1}{r} + \ell(\mathbf{x},r), \tag{6.10}$$

where

$$\ell(\mathbf{x},r) = -\frac{1}{\pi}\mathbb{E}_{\mu^{\mathbf{x},r}}\left[\log\frac{1}{|X_{\tau_D}-\mathbf{x}|}\right]. \tag{6.11}$$

Since the left hand side of (6.10) equals $\langle R\mu^{\mathbf{x},r}, \nu^{\mathbf{x},r}\rangle = f(\mathbf{x},r)$, we obtain

$$f(\mathbf{x},r) = \frac{1}{\pi}\log\frac{1}{r} + \ell(\mathbf{x},r) \quad \mathbf{x} \in D_0, \quad r \in (0.1). \tag{6.12}$$

for $\ell(\mathbf{x},r)$ given by (6.11). By (6.11), $\ell(\mathbf{x},r)$ is bounded in $\mathbf{x} \in D_0$ uniformly in $r \in (0,1)$. Therefore, by (6.12), the condition **(C.5)** is fulfilled.

Property (6.4) now holds with $\kappa = 1$. In order to verify this, take any $\mathbf{x},\mathbf{y} \in D_0$, and any $\varepsilon,\ \delta \in (0,1/2)$ with $1 > |\mathbf{x}-\mathbf{y}| > \varepsilon + \delta$. It then follows from (6.9) that

$$\begin{cases} \langle \mu^{\mathbf{x},\varepsilon}, R\mu^{\mathbf{y},\delta}\rangle = \mathrm{I} - \mathrm{II}, \quad \text{where} \\ \mathrm{I} = \frac{1}{\pi}\int_{\partial B(\mathbf{x},\varepsilon)\times\partial B(\mathbf{y},\delta)} \log\frac{1}{|\mathbf{z}-\mathbf{z}'|}\mu^{\mathbf{x},\varepsilon}(d\mathbf{z})\mu^{\mathbf{y},\delta}(d\mathbf{z}'), \\ \mathrm{II} = \frac{1}{\pi}\int_{\partial D\times\partial B(\mathbf{y},\delta)} \log\frac{1}{|\xi-\mathbf{z}|}\mathbb{P}_{\mu^{\mathbf{x},\varepsilon}}(X_{\tau_D} \in d\xi)\mu^{\mathbf{y},\delta}(d\mathbf{z}). \end{cases} \tag{6.13}$$

As $|\mathbf{z}-\mathbf{z}'| \ge |\mathbf{x}-\mathbf{y}|-\varepsilon-\delta$ for $\mathbf{z}\in \partial B(\mathbf{x},\varepsilon)$ and $\mathbf{z}'\in\partial B(\mathbf{y},\delta)$, we have

$$\mathrm{I} \le \frac{1}{\pi}\log\frac{1}{|\mathbf{x}-\mathbf{y}|-\varepsilon-\delta}. \tag{6.14}$$

Since $\mathrm{dist}(\partial D, \partial B(\mathbf{y},\delta)) \ge 1/2$, and D is bounded, II is uniformly bounded. Hence (6.12) and (6.14) imply that the property (6.4) with $\kappa = 1$ is fulfilled.

Let $\{X_u; u \in H_0^1(D)\}$ be the centered Gaussian field defined on a probability space $(\Omega, \mathcal{B}, \mathbb{P})$ with covariance $\mathbb{E}(X_uX_v) = \frac{1}{2}\mathbf{D}_D(u,v),\ u,v\in H_0^1(D)$. Define $Y^{\mathbf{x},r}$ and $\widetilde{Y}^{\mathbf{x},\varepsilon,\gamma}$ by (6.7). Define $\mu_\varepsilon(A,\omega)$ by (6.8) for a positive finite measure σ on D_0 specified there. Then we can use Theorem 6.2 along with the expression (6.12) of $f(\mathbf{x},r)$ in the same way as in Example 5.1 to get the stated convergence of $\mu_\varepsilon(\cdot,\omega)$ as $\varepsilon\downarrow 0$ for $\gamma\in(0,2\sqrt{\pi})$.

6.2 Transformations of GMCs by conformal maps

Let D and $\widehat{D}$ be domains of $\mathbb{C}$ with $\mathbb{C}\setminus D$ and $\mathbb{C}\setminus\widehat{D}$ being non-polar with respect to the Brownian motion on $\mathbb{C}$, and ψ be a conformal map from D onto $\widehat{D}$. We write

$$\psi(x+iy) = u+iv\in\widehat{D}, \qquad x+iy\in D.$$

We consider the Dirichlet form $\mathcal{E} = (\frac{1}{2}\mathbf{D}_D, H_0^1(D))$ (resp.$\widehat{\mathcal{E}} = (\frac{1}{2}\mathbf{D}_{\widehat{D}}, H_0^1(\widehat{D}))$) on $L^2(D)$ (resp.$L^2(\widehat{D})$) and its extended Dirichlet space $H_{0,e}^1(D)$ (resp.$H_{0,e}^1(\widehat{D})$) as was described preceding to Example 6.3. For a function f on D, define a function Ψf on $\widehat{D}$ by $(\Psi f)(u+iv) = f\circ\psi^{-1}(u+iv),\ u+iv\in\widehat{D}$. We then readily obtain

$$H_{0,e}^1(\widehat{D}) = \{\Psi f : f\in H_{0,e}^1(D)\}, \quad \frac{1}{2}\mathbf{D}_{\widehat{D}}(\Psi f,\Psi g) = \frac{1}{2}\mathbf{D}_D(f,g), \quad f,g\in H_{0,e}^1(D). \tag{6.15}$$

Ψ is a bijection between $H_{0,e}^1(D)$ and $H_{0,e}^1(\widehat{D})$.

We first note the conformal invariance of potential theoretic notions. For any subset A of D, $\mathrm{Cap}^{(0)}(A)$ denotes its 0-order capacity with respect to the Dirichlet form $\mathcal{E}$ and, for $A\in\mathcal{B}(D)$, p_A denotes its hitting probability of the absorbing Brownian motion $\mathbb{M}^D$: $p_A(\mathbf{x}) = \mathbb{P}_{\mathbf{x}}^D(\sigma_A<\infty),\ \mathbf{x}\in D$. The correponding notions for $\widehat{D}$ will be designated with $\widehat{\ }$.

Lemma 6.4 (i) *It holds for any set $A\subset D$ that*

$$\mathrm{Cap}^{(0)}(A) = \widehat{\mathrm{Cap}}^{(0)}(\psi(A)). \tag{6.16}$$

A set $A\subset D$ is $\mathcal{E}$-polar iff $\psi(A)$ is $\widehat{\mathcal{E}}$-polar. A function f on D is $\mathcal{E}$-quasi-continuous iff Ψf is $\widehat{\mathcal{E}}$-quasi-continuous.
(ii) *For any $A\in\mathcal{B}(D)$ with $\mathrm{Cap}^{(0)}(A)<\infty$.*

$$\Psi p_A(\mathbf{y}) = \widehat{p}_{\psi(A)}(\mathbf{y}), \quad \text{for } \widehat{\mathcal{E}}-\text{q.e. } \mathbf{y}\in\widehat{D}. \tag{6.17}$$

Proof. (i). It suffices to show (6.16) for any open set $A\subset D$. Then, $\mathrm{Cap}^{(0)}(A) = \inf\{\frac{1}{2}\mathbf{D}_D(f,f) : f\in H_{0,e}^1(D),\ f\ge 1 \text{ a.e. on } A\}$, which combined with (6.15) yields (6.16).
(ii). By the 0-order version of [FOT, Th.2.1.5] and [FOT, Th.4.3.3], p_A is $\mathcal{E}$-quasi-continuous function in $H_{0,e}^1(D)$ and characterized by the conditions that $p_A = 1$ $\mathcal{E}$-q.e. on A and $\frac{1}{2}\mathbf{D}_D(p_A,v)\ge 0$, for any $v\in H_{0,e}^1(D)$ with $\widetilde{v}\ge 0$ $\mathcal{E}$-q.e. on A, which combined with (i) and (6.15) yields (6.17).
□

For any compact set $B \subset D$ with $\mathrm{Cap}^{(0)}(B) > 0$, the (renormalized) equilibrium potential p^B and the Robin constant $c(B)$ of B are given by $p^B = p_B/\mathrm{Cap}^{(0)}(B)$ and $c(B) = 1/\mathrm{Cap}^{(0)}(B)$, respectively. The above lemma implies

$$\Psi p^B = \widehat{p}^{\psi(B)}, \quad c(B) = \widehat{c}(\psi(B)). \tag{6.18}$$

Let $\mathbb{G}(\mathcal{E}) = \{X_f; f \in H^1_{0,e}(D)\}$ be the centered Gaussian field indexed by $H^1_{0,e}(D)$ defined on a probability space $(\Omega, \mathcal{B}, \mathbb{P})$ with $\mathbb{E}[X_f X_g] = \frac{1}{2}\mathbf{D}_D(f,g),\ f,g \in H^1_{0,e}(D)$. For any $\widehat{f} \in H^1_{0,e}(\widehat{D})$, $\Psi^{-1}\widehat{f} \in H^1_{0,e}(D)$ by (6.15) so that one may define the random variable

$$\widehat{X}_{\widehat{f}} = X_{\Psi^{-1}\widehat{f}}. \tag{6.19}$$

(6.15) then implies that $\mathbb{G}(\widehat{\mathcal{E}}) = \{\widehat{X}_{\widehat{f}} : \widehat{f} \in H^1_{0,e}(\widehat{D})\}$ becomes a centered Gaussian field indexed by $H^1_{0,e}(\widehat{D})$ with $\mathbb{E}[\widehat{X}_{\widehat{f}}\widehat{X}_{\widehat{g}}] = \frac{1}{2}\mathbf{D}_{\widehat{D}}(\widehat{f},\widehat{g}),\ \widehat{f},\widehat{g} \in H^1_{0,e}(\widehat{D})$. It further follows from (6.18) that, for any compact set $B \subset D$ with $\mathrm{Cap}^{(0)}(B) > 0$,

$$\widehat{X}_{\widehat{p}^{\psi(B)}} = X_{p^B}. \tag{6.20}$$

We now consider a bounded open set D_0 with $\overline{D}_0 \subset D$ and choose, for each $\mathbf{x} \in D_0$, a family $\{B(\mathbf{x},\varepsilon), \varepsilon > 0\}$ of compact sets with

$$\mathbf{x} \in B^o(\mathbf{x},\varepsilon),\ B(\mathbf{x},\varepsilon) \subset D,\ \mathrm{Cap}^{(0)}(B(\mathbf{x},\varepsilon)) > 0\ \forall\, \varepsilon > 0, \quad \text{and} \quad B(\mathbf{x},\varepsilon) \downarrow \{\mathbf{x}\} \text{ as } \varepsilon \downarrow 0,$$

where $B^o(\mathbf{x},\varepsilon)$ denotes the interior of $B(\mathbf{x},\varepsilon)$. We assume that for each $\varepsilon > 0$

$$\text{the map} \quad \mathbf{x} \in D_0 \quad \text{to } p^{B(\mathbf{x},\varepsilon)} \in H^1_{0,e}(D) \quad \text{is continuous.} \tag{6.21}$$

Define

$$Y^{\mathbf{x},\varepsilon} = X_{p^{B(\mathbf{x},\varepsilon)}}, \quad f(\mathbf{x},\varepsilon) = 1/\mathrm{Cap}^{(0)}(B(\mathbf{x},\varepsilon)), \quad \mathbf{x} \in D_0, \varepsilon > 0. \tag{6.22}$$

We fix a constant $\gamma > 0$. Given a finite positive measure σ on D_0, we introduce a random measure $\mu_\varepsilon(\cdot,\omega)$ on D_0 by

$$\mu_\varepsilon(A,\omega) = \int_A \exp\left(\gamma Y^{\mathbf{x},\varepsilon} - \frac{\gamma^2}{2} f(\mathbf{x},\varepsilon)\right) \sigma(d\mathbf{x}), \quad A \in \mathcal{B}(D_0). \tag{6.23}$$

Here $Y^{\mathbf{x},\varepsilon}$ in (6.22) is chosen to be its measurable version of $\mathbf{x}$, namely, a function $Y(\mathbf{x},\omega)$ measurable in $(\mathbf{x},\omega) \in D_0 \times \Omega$ such that, for σ-a.e. $\mathbf{x} \in D_0$, $Y(\mathbf{x},\omega) = Y^{\mathbf{x},\varepsilon}(\omega)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$. The existence of such a version is ensured by the assumption (6.21) as in the proof of Proposition 4.9.

Let $\mathcal{M}(D_0)$ be the space of all finite positive measures on D_0 equipped with the topology of the weak convergence, which can be induced by the metric $\rho(\mu,\nu),\ \mu,\nu \in \mathcal{M}(D_0)$, defined by (4.48) using any countable dense subfamily $\{g_n\}$ of $C(\overline{D}_0)$.

Notice that $\mu_\varepsilon(\cdot,\omega) \in \mathcal{M}(D_0)$ for almost all $\omega \in \Omega$. If there exists $\nu(\cdot,\omega) \in \mathcal{M}(D_0)$ such that $\lim_{\varepsilon\downarrow 0} \mathbb{P}(\rho(\mu_\varepsilon,\nu) > \delta) = 0$ for any $\delta > 0$, then we say that the Gaussian field $\mathbb{G}(\mathcal{E})$ *admits a multiplicative chaos* $\nu(\cdot,\omega)$ *on* D_0 *relative to* σ *and* $\{B(\mathbf{x},\varepsilon),\ \mathbf{x} \in D_0,\ \varepsilon > 0\}$.

Put $\widehat{D}_0 = \psi(D_0)$. Let us define, for $\mathbf{y} \in \widehat{D}_0,\ \varepsilon > 0$,

$$\widehat{B}(\mathbf{y},\varepsilon) = \psi(B(\psi^{-1}\mathbf{y},\varepsilon)), \quad \widehat{Y}^{\mathbf{y},\varepsilon} = \widehat{X}_{\widehat{p}^{\widehat{B}(\mathbf{y},\varepsilon)}}, \quad \widehat{f}(\mathbf{y},\varepsilon) = 1/\mathrm{Cap}^{(0)}(\widehat{B}(\mathbf{y},\varepsilon)). \tag{6.24}$$

$\{\widehat{B}(\mathbf{y},\varepsilon) : \mathbf{y} \in \widehat{D}_0, \varepsilon > 0\}$ is then a family of compact subsets of $\widehat{D}_0$ containing $\mathbf{y}$ in the interior with $\widehat{B}(\mathbf{y},\varepsilon) \downarrow \{\mathbf{y}\}$ as $\varepsilon \downarrow 0$. We further let

$$\widehat{\sigma} = \psi \cdot \sigma : \quad \widehat{\sigma}(C) = \sigma(\psi^{-1}(C)), \quad C \in \mathcal{B}(\widehat{D}_0). \tag{6.25}$$

For instance, when σ is the Lebegue measure dx_1dx_2 on D_0, $\widetilde{\sigma}(dy_1dy_2) = \frac{1}{|\psi'(\mathbf{y})|^2}\, dy_1dy_2$.

Theorem 6.5 *Let ψ be a conformal map from D onto $\widehat{D}$. $\mathbb{G}(\mathcal{E})$ admits a multiplicative chaos ν on D_0 relative to σ and $\{B(\mathbf{x},\varepsilon), \mathbf{x}\in D_0,\ \varepsilon>0\}$ if and only if $\mathbb{G}(\widehat{\mathcal{E}})$ admits a multiplicative chaos $\widehat{\nu}$ on $\widehat{D}_0=\psi(D_0)$ relative to $\widehat{\sigma}$ defined by (6.25) and $\{\widehat{B}(\mathbf{y},\varepsilon), \mathbf{y}\in \widehat{D}_0,\ \varepsilon>0\}$ defined by (6.24).*

In this case, $\widehat{\nu}$ is the image measure of ν by ψ: $\widehat{\nu}(C)\ =\nu(\psi^{-1}(C)),\ C\in\mathcal{B}(\widehat{D}_0)$.

Proof. It follows from (6.20) and (6.24) that

$$\widehat{Y}^{\mathbf{y},\varepsilon}=Y^{\psi^{-1}\mathbf{y},\varepsilon},\quad \widehat{f}(\mathbf{y},\varepsilon)=f(\psi^{-1}\mathbf{y},\varepsilon),\quad \mathbf{y}\in\widehat{D},\quad \varepsilon>0. \tag{6.26}$$

Therefore

$$\begin{aligned}\langle g_n,\mu_\varepsilon\rangle &= \int_{D_0} g_n(\mathbf{x})\exp\left(\gamma\widehat{Y}^{\mathbf{y},\varepsilon}-\frac{\gamma^2}{2}\widehat{f}(\mathbf{y},\varepsilon)\right)\Big|_{\mathbf{y}=\psi(\mathbf{x})}\sigma(d\mathbf{x})\\ &= \int_{\widehat{D}_0}\Psi g_n(\mathbf{y})\exp\left(\gamma\widehat{Y}^{\mathbf{y},\varepsilon}-\frac{\gamma^2}{2}\widehat{f}(\mathbf{y},\varepsilon)\right)\ \widehat{\sigma}(d\mathbf{y})=\langle\Psi g_n,\widehat{\mu}_\varepsilon\rangle.\end{aligned}$$

We also have $\langle g_n,\nu\rangle=\langle\Psi g_n,\widehat{\nu}\rangle$ for the image measure $\widehat{\nu}$ of ν by ψ. The assertion of theorem follows by noting that $\{\Psi g_n,\ n\ge 1\}$ is dense in the space of all continuous functions on the closure of $\widehat{D}_0$. □

7 Appendix

7.1 Proof of Proposition 4.7

Proof of Proposition 4.7 (i)

Using the function $\check{r}_1(\mathbf{x},\mathbf{y}),\ \mathbf{x},\mathbf{y}\in F$, in (4.13), define

$$\check{r}_1^1(\mathbf{x},\mathbf{y})=\check{r}_1(\mathbf{x},\mathbf{y}),\quad \check{r}_1^n(\mathbf{x},\mathbf{y})=\int\check{r}_1(\mathbf{x},\mathbf{z})\check{r}_1^{n-1}(\mathbf{z},\mathbf{y})m_F(d\mathbf{z}),\ n\ge 2.$$

$\check{r}_1^n(\mathbf{x},\mathbf{y})$ is the density function of the kernel $\check{R}^n(\mathbf{x},d\mathbf{y})$ on $(F,\mathcal{B}(E))$ with respect to m_F. $\check{R}_1^n(\mathbf{x},d\mathbf{y})$ is m_F-symmetric and $\check{R}_1^n 1_F(\mathbf{x})=1,\ \mathbf{x}\in F$, so that $\widetilde{m}_F\check{R}_1^n=\widetilde{m}_F$. Consequently,

$$\check{R}_1^n(\mathbf{x},A)-\widetilde{m}_F(A)=\int_F[\check{R}_1^n(\mathbf{x},A)-\check{R}_1^n(\mathbf{x}',A)]\widetilde{m}_F(d\mathbf{x}'),\quad A\in\mathcal{B}(F).$$

Denote by $||\mu||$ the total variation of a signed measure μ on F, We then get from the above identity and an estimate [FO, (3.4)]

$$\sup_{\mathbf{x}\in F}||\check{R}_1^n(\mathbf{x},\cdot)-\widetilde{m}_F(\cdot)||\le 2\gamma^n,\quad \text{for some constant } \gamma\in(0,1). \tag{7.1}$$

Therefore, if we let

$$\check{r}^{(\pm)}(\mathbf{x},A)=\int_A\sum_{n=1}^{\infty}(\check{r}_1^n(\mathbf{x},\mathbf{y})-1/m(F))^{\pm}m_F(d\mathbf{y}),\ \mathbf{x}\in F,\ A\in\mathcal{B}(F), \tag{7.2}$$

then, $\check{r}^{(+)}(\mathbf{x},A),\ \check{r}^{(-)}(\mathbf{x},A)$ are positive kernels on $(F,\mathcal{B}(F))$ satisfying $\sup_{\mathbf{x}\in F}\check{r}^{(\pm)}(\mathbf{x},F)<\infty$ and, for any $\varphi\in L^\infty(F;m_F)$,

$$\check{R}\varphi(\mathbf{x})=\int_F\check{r}^{(+)}(\mathbf{x},d\mathbf{y})\varphi(\mathbf{y})-\int_F\check{r}^{(-)}(\mathbf{x},d\mathbf{y})\varphi(\mathbf{y})\ \text{for } m\text{−a.e. } \mathbf{x}\in F. \tag{7.3}$$

on account of (4.14). This identity can be readily verified to hold also for $\varphi \in L^2(F;m_F)$.

Define $r^{(\pm)}(\mathbf{x},\mathbf{y}) = \int_F \int_{F\times F} r^g(\mathbf{x},\mathbf{z})\check{r}^{(\pm)}(\mathbf{z},d\mathbf{w})r^g(\mathbf{w},\mathbf{y})m_F(d\mathbf{z})$ for $\mathbf{x},\mathbf{y} \in \mathbb{C}$, $r^{(\pm)}(\mathbf{x},\mathbf{y})$ are symmetric and $\mathbb{M}^g$-excessive for each variable $\mathbf{x}$ and $\mathbf{y}$. $\int_{\mathbb{C}} r^{(+)}(\mathbf{x},\mathbf{y})h(\mathbf{y})m(d\mathbf{y})$ is finite for each $\mathbf{x}\in\mathbb{C}$ for any non-negative bounded Borel function h on $\mathbb{C}$ vanishing outside a bounded set, because $R^g h$ is bounded on $\mathbb{C}$ by Lemma 3.1 and so $\psi(\mathbf{z}) = \int_F \check{r}^{(+)}(\mathbf{z},d\mathbf{w})R^g h(\mathbf{w})$ is bounded on F by a constant $C>0$, and furthermore $\int_{\mathbb{C}} r^{(+)}(\mathbf{x},\mathbf{y})h(\mathbf{y})m(d\mathbf{y}) = R^g(1_F\cdot\psi)(\mathbf{x}) \le CR^g g(\mathbf{x}) = C$ in view of [FO, (3.28)]. Therefore $r^{(\pm)}(\mathbf{x},\mathbf{y})$ is finite for m-a.e. $\mathbf{y}$ and hence q.e. $\mathbf{y}\in\mathbb{C}$.

We see from (4.14) that, for $\varphi \in L^2(F;m_F)$, $\check{R}\varphi = \check{R}_1\varphi - \langle \widetilde{m}_F,\varphi\rangle + \check{R}_1\check{R}\varphi$. Consider any $\mu \in \mathcal{S}_0^{g,(0)}$ with $\mu(\mathbb{C})<\infty$. Since $R^g\mu \in L^2(\mathbb{C},m_F)$ and $\langle \widetilde{m}_F, R^g\mu\rangle = \frac{1}{m(F)}\langle R^g g,\mu\rangle = \mu(\mathbb{C})/m(F)$, we have

$$H_F\check{R}(1_F R^g\mu)(\mathbf{x}) = R^g(1_F R^g\mu)(\mathbf{x}) - \mu(\mathbb{C})/m(F) + R^g(1_F\check{R}(1_F R^g\mu))(\mathbf{x}), \quad \mathbf{x}\in\mathbb{C},$$

which combined with (4.12) and (7.3) implies that $R\mu$ admits an expression (4.26) by a kernel $\mathfrak{r}(\mathbf{x},\mathbf{y})$ defined by

$$\mathfrak{r}(\mathbf{x},\mathbf{y}) = \int_F r^g(\mathbf{x},\mathbf{z})r^g(\mathbf{z},\mathbf{y})m_F(d\mathbf{z}) + r^{(+)}(\mathbf{x},\mathbf{y}) - r^{(-)}(\mathbf{x},\mathbf{y}) + r^g(\mathbf{x},\mathbf{y}) - 2/m(F), \quad \mathbf{x},\mathbf{y}\in\mathbb{C}. \tag{7.4}$$

$\mathfrak{r}(\mathbf{x},\mathbf{y})$ is symmetric and, for each $\mathbf{x}\in\mathbb{C}$, it is a difference of M^g-excessive functions finite for q.e. $\mathbf{y}\in\mathbb{C}$. This property for the first term of the righthand side can be verified in a similar way to the proof for other terms given previously. □

Proof of Proposition 4.7 (ii)

We take $\mathbf{x},\mathbf{y}\in B(S-1)$ with $|\mathbf{x}-\mathbf{y}|>\eta$. Since there exists a constant M_2 such that $|\widehat{R}\mu^{\mathbf{y},r_2}| \le M_2$ on $B(\mathbf{x},\eta/8)$ for any $r_2<\eta/8$ by Lemma 4.6, the stated uniform boundedness of $\langle \mu^{\mathbf{x},r_1}, R\mu^{\mathbf{y},r_2}\rangle = \langle \mu^{\mathbf{x},r_1}, \widehat{R}\mu^{\mathbf{y},r_2}\rangle$ holds true. To prove (4.27), we first show that

$$\lim_{r_1,r_2\downarrow 0}\langle \mu^{\mathbf{x},r_1}, R^g\mu^{\mathbf{y},r_2}\rangle = r^g(\mathbf{x},\mathbf{y}), \tag{7.5}$$

for $m\times m$-a.e. $(\mathbf{x},\mathbf{y})\in B(S-1)\times B(S-1)\cap\{(\mathbf{x},\mathbf{y}): |\mathbf{x}-\mathbf{y}|>\eta\}$.

Since $e^{-t}p_t(\mathbf{z},\mathbf{w}) \le p_t^g(\mathbf{z},\mathbf{w}) \le p_t(\mathbf{z},\mathbf{w})$, we can use (4.1) to find for any $\varepsilon>0$ a positive t_0 satisfying

$$\int_0^t p_s^g(\mathbf{z},\mathbf{w})ds \le \int_0^t \frac{K_2}{s}e^{-9k_2\eta^2/16s}ds < \varepsilon \tag{7.6}$$

for any $t\le t_0$ and $\mathbf{z},\mathbf{w}\in E$ such that $|\mathbf{z}-\mathbf{w}|>3\eta/4$. In particular, $\int_0^t\langle \mu^{\mathbf{x},r_1}, P_s^g\mu^{\mathbf{y},r_2}\rangle ds<\varepsilon$.

Let M_1 be a constant satisfying $R^g\mu^{\mathbf{y},r_2}\le M_1$ on $\mathbb{C}\setminus B(\mathbf{y},\eta/2)$ for all $r_2\le\eta/8$. Such constant M_1 exists by Lemma 4.6. By (4.1) and the tail estimate (4.32), we may assume that, by taking smaller $t_0>0$ if necessary,

$$\int_{\mathbb{C}\setminus B(\mathbf{x},\eta/2)} p_t^g(\mathbf{z},\mathbf{w})m(d\mathbf{w}) \le \int_{\{|\mathbf{w}|>3\eta/8\}}\frac{K_2}{t}e^{-k_2|\mathbf{w}|^2/t}d\mathbf{w} \le K_2\sqrt{\pi/k_2 t}\,e^{-9\eta^2k_2/64t} < \frac{\varepsilon}{M_1}$$

for all $t\le t_0$ and $\mathbf{z}\in B(\mathbf{x},\eta/8)$. In particular,

$$\begin{aligned}\langle \mu^{\mathbf{x},r_1}P_t^g, 1_{\mathbb{C}\setminus B(S-1/2)}R^g\mu^{\mathbf{y},r_2}\rangle &= \int\int p_t^g(\mathbf{z},\mathbf{w})1_{\mathbb{C}\setminus B(S-1/2)}(\mathbf{w})R^g\mu^{\mathbf{y},r_2}(\mathbf{w})m(d\mathbf{w})\mu^{\mathbf{x},r_1}(d\mathbf{z})\\ &\le M_1\int_{B(\mathbf{x},\eta/8)}\mu^{\mathbf{x},r_1}(d\mathbf{z})\int_{\mathbb{C}\setminus B(\mathbf{x},\eta/2)}p_t^g(\mathbf{z},\mathbf{w})m(d\mathbf{w}) < \varepsilon. \end{aligned}\tag{7.7}$$

Put $D(\mathbf{y}) = B(S-1/2)\setminus \overline{B(\mathbf{y},\eta/2)}$. Since $p_t^g(\mathbf{z},\mathbf{w}) \le (K_2/t)e^{-9k_2\eta^2/64t}$ for any $\mathbf{z}\in B(\mathbf{x},\eta/8)$ and $\mathbf{w}\in B(\mathbf{y},\eta/2)$ and $R^g 1_{B(S-1)} \le M_4$ on $\mathbb{C}$ for some constant M_4 by Lemma 3.1 (i),

$$\langle \mu^{\mathbf{x},r_1}P_t^g, 1_{B(\mathbf{y},\eta/2)}\cdot R^g\mu^{\mathbf{y},r_2}\rangle \le \frac{K_2}{t}e^{-9k_2\eta^2/64t}\langle \mu^{\mathbf{y},r_2}, R^g 1_{B(\mathbf{y},\eta/2)}\rangle \le \frac{K_2M_4}{t}e^{-9k_2\eta^2/64t} < \varepsilon \quad (7.8)$$

for any $t<t_0$ by taking smaller t_0 if necessary. Further, since the distance between F and $B(\mathbf{x},\eta/8)$ exceeds $1/2$, we get by putting $A_t = \int_0^t 1_F(X_s)ds$,

$$\begin{aligned} P_{t_0}(\mathbf{z},D(\mathbf{y})) - P_{t_0}^g(\mathbf{z},D(\mathbf{y})) &= \mathbb{E}_{\mathbf{z}}\left[(1-e^{-A_{t_0}})1_{D(\mathbf{y})}(X_{t_0})\right] \le \mathbb{E}_{\mathbf{z}}\left[\int_0^{t_0} 1_F(X_s)ds\right] \\ &\le \int_0^{t_0}\frac{K_2 m(F)}{s}e^{-k_2/4s}ds \quad \text{for any } \mathbf{z}\in B(\mathbf{x},\eta/8). \end{aligned}$$

Hence we may also assume that

$$\langle \mu^{\mathbf{x},r_1}, (P_t-P_t^g)(1_D R^g\mu^{\mathbf{y},r_2})\rangle < \varepsilon \quad \text{for all } t\le t_0, \quad (7.9)$$

because $R^g\mu^{\mathbf{x},r_2}\le M_1$ on $D(\mathbf{y})$.

Therefore, in the decomposition

$$\begin{aligned} \langle \mu^{\mathbf{x},r_1}, R^g\mu^{\mathbf{y},r_2}\rangle &= \int_0^t \langle \mu^{\mathbf{x},r_1}, P_s^g\mu^{\mathbf{y},r_2}\rangle ds + \langle \mu^{\mathbf{x},r_1}, P_t^g R^g\mu^{\mathbf{y},r_2}\rangle \\ &= \int_0^t \langle \mu^{\mathbf{x},r_1}, P_s^g\mu^{\mathbf{y},r_2}\rangle ds + \langle \mu^{\mathbf{x},r_1}P_t^g, 1_{\mathbb{C}\setminus B(S-1/2)}\cdot R^g\mu^{\mathbf{y},r_2}\rangle \\ &\quad + \langle \mu^{\mathbf{x},r_1}P_t^g, 1_{B(\mathbf{y},\eta/2)}\cdot R^g\mu^{\mathbf{y},r_2}\rangle + \langle \mu^{\mathbf{x},r_1}, (P_t^g-P_t)(1_{D(\mathbf{y})}R^g\mu^{\mathbf{y},r_2})\rangle \\ &\quad + \langle \mu^{\mathbf{x},r_1}, P_t(1_{D(\mathbf{y})}R^g\mu^{\mathbf{y},r_2})\rangle, \end{aligned}$$

the sum of the first four terms of the righthand side is smaller than 4ε for any $r_1,r_2\in(0,\eta/8)$ and $t\le t_0$.

Since $p_t(\mathbf{z},\mathbf{w})$ is uniformly continuous relative to $(\mathbf{z},\mathbf{w})$ on $\overline{B(\mathbf{x},\eta/8)}\times\overline{D}(\mathbf{y})$, by putting $\delta(t,r_1) = \sup\{|p_t(\mathbf{z},\mathbf{w})-p_t(\mathbf{x},\mathbf{w})| : \mathbf{z}\in\overline{B(\mathbf{x},r_1)}, \mathbf{w}\in\overline{D}(\mathbf{y})\}$, we can see that the difference of the last term of the righthand side and $\int_{D(\mathbf{y})} p_t(\mathbf{x},\mathbf{w})R^g\mu^{\mathbf{y},r_2}(\mathbf{w})m(d\mathbf{w})$ is smaller than $M_1\delta(t,r_1)$ which converges to zero as $r_1\downarrow 0$ for each $t<t_0$. Furthermore, for $f_t^{\mathbf{x}}(\mathbf{w}) = 1_{D(\mathbf{y})}(\mathbf{w})p_t(\mathbf{x},\mathbf{w})$, $R^g f_t^{\mathbf{x}}$ is $\mathcal{E}$-harmonic on $B(\mathbf{y},\eta/8)$ by Lemma 3.1 and continuous there as in the proof of Lemma 4.4. Consequently, $\lim_{r_2\to 0}\int_{D(\mathbf{y})} p_t(\mathbf{x},\mathbf{w})R^g\mu^{\mathbf{y},r_2}(\mathbf{w})m(d\mathbf{w}) = \lim_{r_2\to 0}\langle \mu^{\mathbf{y},r_2}, R^g f_t^{\mathbf{x}}\rangle = R^g f_t^{\mathbf{x}}(\mathbf{y})$. Accordingly

$$\limsup_{r_1,r_2\downarrow 0} |\langle \mu^{\mathbf{x},r_1}, R^g\mu^{\mathbf{y},r_2}\rangle - R^g f_t^{\mathbf{x}}(\mathbf{y})| < 4\varepsilon \quad (7.10)$$

for any $t\le t_0$ and any $\mathbf{x},\mathbf{y}\in B(S-1)$ with $|\mathbf{x}-\mathbf{y}|>\eta$.

Thus, to verify (7.5), it suffices to show that $\lim_{t\to 0} R^g f_t^{\mathbf{x}}(\mathbf{y}) = \lim_{t\to 0} P_t(1_{D(\mathbf{y})}\cdot r^g(\cdot,\mathbf{y}))(\mathbf{x}) = r^g(\mathbf{x},\mathbf{y})$ for $m\times m$-a.e. $(\mathbf{x},\mathbf{y})\in B(S-1)\times B(S-1)\cap\{|\mathbf{x}-\mathbf{y}|>\eta\}$. For any $\mathbf{y}\in B(S-1)$, let $E_1(\mathbf{y}) = \{\mathbf{x} : r^g(\mathbf{x},\mathbf{y})<\infty\}$. As $\mathbb{C}\setminus E_1(\mathbf{y})$ is polar and

$$P_t^g(1_{D(\mathbf{y})}r^g(\cdot,\mathbf{y}))(\mathbf{x}) \le P_t(1_{D(\mathbf{y})}r^g(\cdot,\mathbf{y}))(\mathbf{x}) \le e^t P_t^g(1_{D(\mathbf{y})}r^g(\cdot,\mathbf{y}))(\mathbf{x}),$$

it is enough to show that $\lim_{t\to 0} P_t^g(1_{D(\mathbf{y})}\cdot r^g(\cdot,\mathbf{y}))(\mathbf{x}) = r^g(\mathbf{x},\mathbf{y})$ for any $\mathbf{x}\in D(\mathbf{y})\cap E_1(\mathbf{y})$. Since $r^g(\cdot,\mathbf{y})$ is $\mathbb{M}^g$-excessive and $1_{D(\mathbf{y})}(X_t)r^g(X_t,\mathbf{y})$ is right continuous at $t=0$ a.s. $\mathbb{P}_{\mathbf{x}}$ for $\mathbf{x}\in D(\mathbf{y})\cap E_1(\mathbf{y})$, we have

$$\begin{aligned} r^g(\mathbf{x},\mathbf{y})\wedge n &= \lim_{t\to 0}\mathbb{E}_{\mathbf{x}}^g\left[1_{D(\mathbf{y})}(X_t)r^g(X_t,\mathbf{y})\wedge n\right] \le \underline{\lim}_{t\to 0}\mathbb{E}_{\mathbf{x}}^g\left[1_{D(\mathbf{y})}(X_t)r^g(X_t,\mathbf{y})\right] \\ &\le \overline{\lim}_{t\to 0}\mathbb{E}_{\mathbf{x}}^g\left[1_{D(\mathbf{y})}(X_t)r^g(X_t,\mathbf{y})\right] \le \lim_{t\to 0}\mathbb{E}_{\mathbf{x}}^g\left[r^g(X_t,\mathbf{y})\right] = r^g(\mathbf{x},\mathbf{y}), \quad n\ge 1. \end{aligned}$$

By letting $n \to \infty$, we arrive at (7.5).

We shall next show that, for the kernels $r^+(\mathbf{x},\mathbf{y})$ and $r^-(\mathbf{x},\mathbf{y})$ appearing in the proof of Proposition 4.7 (i),

$$\lim_{r_1,r_2\downarrow 0}\langle \mu^{\mathbf{x},r_1}, R^{(\pm)}\mu^{\mathbf{y},r_2}\rangle = r^{(\pm)}(\mathbf{x},\mathbf{y}), \tag{7.11}$$

for $m\times m$-a.e. $(\mathbf{x},\mathbf{y})\in B(S-1)\times B(S-1)$. Here we let $R^{(\pm)}\mu(\mathbf{x}) = \int_{\mathbb{C}} r^{(\pm)}(\mathbf{x},\mathbf{z})\mu(d\mathbf{z}),\ \mathbf{x}\in\mathbb{C}$. Consider the function on $\mathbb{C}$ defined by $\xi_+^{\mathbf{y},r_2}(\mathbf{z}) = 1_F(\mathbf{z})\check{R}^{(+)}(1_F R^g\mu^{\mathbf{y},r_2})(\mathbf{z}),\ \mathbf{z}\in\mathbb{C}$. Since $R^g\mu^{\mathbf{y},r_2}(\mathbf{z})$ is bounded in $\mathbf{z}\in F$ and r_2 by Lemma 4.6 (i) and $\check{R}^{(+)}$ is a bounded linear operator on $L^\infty(F;m_F)$, there exists a constant $M>0$ such that for any $\mathbf{z}\in\mathbb{C},\ r_2\in(0,\eta/8)$,

$$\xi_+^{\mathbf{y},r_2}(\mathbf{z})\le M,\ R^g\xi_+^{\mathbf{y},r_2}(\mathbf{z}) = \int_F r^g(\mathbf{z},\mathbf{w})\xi_+^{\mathbf{y},r_2}(\mathbf{w})m(d\mathbf{w}) = H_F(R^g\xi_+^{\mathbf{y},r_2})(\mathbf{z})\le M. \tag{7.12}$$

In view of definition, we have the identity $R^{(+)}\mu^{\mathbf{y},r_2} = R^g\xi_+^{\mathbf{y},r_2}$. Accordingly, as in the previous proof of (7.5), we can decompose $\langle\mu^{\mathbf{x},r_1}, R^{(+)}\mu^{\mathbf{y},r_2}\rangle$ as

$$\begin{aligned}\langle\mu^{\mathbf{x},r_1}, R^{(+)}\mu^{\mathbf{y},r_2}\rangle &= \int_0^t \langle\mu^{\mathbf{x},r_1}, P_s^g\xi_+^{\mathbf{y},r_2}\rangle ds + \langle\mu^{\mathbf{x},r_1}P_t^g, 1_{\mathbb{C}\setminus B(S-1/2)}\cdot R^g\xi_+^{\mathbf{y},r_2}\rangle \\ &\quad +\langle\mu^{\mathbf{x},r_1}, (P_t^g-P_t)(1_{B(S-1/2)}R^g\xi_+^{\mathbf{y},r_2})\rangle + \langle\mu^{\mathbf{x},r_1}, P_t(1_{B(S-1/2)}R^g\xi_+^{\mathbf{y},r_2})\rangle.\end{aligned}$$

For any $\varepsilon>0$, we can take t_1 such that the first term of the righthand side is less than ε for any $t\in(0,t_1)$ as (7.6) because of $\mathrm{dist}(F,B(\mathbf{x},r_1))>1/2$ and the bound (7.12). Because also of the bound (7.12), we can take t_1 such that the second term is less than ε for any $t\in(0,t_1)$ as (7.7). Further, as (7.9), we may suppose that the third term is less than ε for all $t\le t_1$.

Since $\sup\{|p_t(\mathbf{z},\mathbf{w})-p_t(\mathbf{x},\mathbf{w})| : \mathbf{z}\in B(\mathbf{x},r_1), \mathbf{w}\in B(S-1/2)\}\to 0$ as $r_1\to 0$, $\lim_{r_1\to 0}\langle\mu^{\mathbf{x},r_1}, P_t(1_{B(S-1/2)}R^g\xi_+^{\mathbf{y},r_2})\rangle = P_t(1_{B(S-1/2)}R^g\xi_+^{\mathbf{y},r_2})(\mathbf{x})$ uniformly in $r_2\le\eta/8$. Put $h_t^{\mathbf{x}}(\mathbf{w}) = 1_F(\mathbf{w})\check{R}^{(+)}R^g(1_{B(S-1/2)}p_t(\cdot,\mathbf{x}))(\mathbf{w})$. Since $h_t^{\mathbf{x}}$ vanishes outside of F, we can see as before that $R^g h_t^{\mathbf{x}}(\mathbf{w})$ is continuous on $B(S-1)$ and consequently

$$\lim_{r_2\to 0} P_t(1_{B(S-1/2)}R^g\xi_+^{\mathbf{y},r_2})(\mathbf{x}) = \lim_{r_2\to 0}\langle\mu^{\mathbf{y},r_2}, R^g h_t^{\mathbf{x}}\rangle = R^g h_t^{\mathbf{x}}(\mathbf{y}).$$

Therefore, as (7.10), $\limsup_{r_1,r_2\downarrow 0}|\langle\mu^{\mathbf{x},r_1}, R^{(+)}\mu^{\mathbf{y},r_2}\rangle - R^g h_t^{\mathbf{x}}(\mathbf{y})| < 3\varepsilon$ for any $t\le t_1$.

As $R^g h_t^{\mathbf{x}}(\mathbf{y}) = R^{(+)}(1_{B(S-1/2)}p_t(\cdot,\mathbf{x}))(\mathbf{y}) = P_t(1_{B(S-1/2)}\cdot r^{(+)}(\cdot,\mathbf{y}))(\mathbf{x})$, and $r^{(+)}(\cdot,\mathbf{y})$ is $\mathbb{M}^g$-excessive and finite q.e., we obtain similarly to the above proof of (7.5), that $\lim_{t\to 0} R^g h_t^{\mathbf{x}}(\mathbf{y}) = r^{(+)}(\mathbf{x},\mathbf{y})$ for q.e. $\mathbf{x}\in B(S-1)$ for each $\mathbf{y}\in B(S-1)$, and consequently, the validity of (7.11) for $R^{(+)}$ and $r^{(+)}$. In the same way (7.11) for $R^{(-)}$ and $r^{(-)}$ is valid.

It remains to prove

$$\lim_{r_1,r_2\downarrow 0}\langle\mu^{\mathbf{x},r_1}, Q\mu^{\mathbf{y},r_2}\rangle = q(\mathbf{x},\mathbf{y}), \tag{7.13}$$

for $m\times m$-a.e. $(\mathbf{x},\mathbf{y})\in B(S-1)\times B(S-1)$. Here $q(\mathbf{x},\mathbf{y})$ is the first term of the righthand side of (7.4) and $Q\mu(\mathbf{x}) = \int_{\mathbb{C}} q(\mathbf{x},\mathbf{z})\mu(z),\ \mathbf{z}\in\mathbb{C}$. But this can be shown in exactly the same way as the proof of (7.11) using $1_F(\mathbf{z})R^g\mu^{\mathbf{y},r_2}(\mathbf{z})$ in place of $\xi_+^{\mathbf{y},r_2}(\mathbf{z})$.

7.2 Proof of Proposition 5.4

Assume that $(a_{ij}(\mathbf{x}))$ is a family of C^1 functions on $\mathbb{C}$ with Hölder continuous derivative satisfying (1.3). Let $b_i(\mathbf{x}) = \sum_{i,j=1}^2 \partial a_{ij}(\mathbf{x})/\partial x_j$ and L be the infinitesimal generator corresponding to the form $\mathbf{a}$:

$$Lu(\mathbf{x}) = \sum_{i,j=1}^2 \frac{\partial}{\partial x_i}\left(a_{ij}(\mathbf{x})\frac{\partial u}{\partial x_j}\right) = \sum_{i,j=1}^2 a_{ij}(\mathbf{x})\frac{\partial^2 u}{\partial x_i\partial x_j} + \sum_{i=1}^2 b_i(\mathbf{x})\frac{\partial u}{\partial x_i}.$$

Let us fix an open disk G containing $\overline{B(S+1)}$. A function $\Gamma(\mathbf{x},\mathbf{y})$ is said to be a *fundamental solution* of L on G if it satisfies $-L\Gamma(\mathbf{x},\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y})$ weakly, that is, for all $u \in C_c^1(G)$,

$$\int_G \sum_{i,j=1}^2 a_{ij}(\mathbf{x}) \frac{\partial u(\mathbf{x})}{\partial x_i} \frac{\partial \Gamma(\mathbf{x},\mathbf{y})}{\partial x_j} d\mathbf{x} = u(\mathbf{y}), \quad \forall \mathbf{y} \in G. \tag{7.14}$$

For any fixed $\mathbf{y} \in G$, let $L_0 u(\mathbf{x}) = \sum_{i,j=1}^2 a_{ij}(\mathbf{y}) \frac{\partial^2 u}{\partial x_i \partial x_j}$. Then $\Gamma_0(\mathbf{x},\mathbf{y})$ defined by (5.19) is a fundamental solution of L_0 on G. We shall briefly describe a construction of a fundamental solution of L from the parametrix $\Gamma_0(\mathbf{x},\mathbf{y})$ as is stated in [Fr, §5.6] under the condition that the coefficients of L are Hölder continuous.

Since $a_{ij} \in C_b^1(G)$, the function $k_0(\mathbf{x},\mathbf{y}) = (L-L_0)\Gamma_0(\mathbf{x},\mathbf{y})$ satisfies, for some constant $K_1 > 0$, $|k_0(\mathbf{x},\mathbf{y})| \le K_1/|\mathbf{x}-\mathbf{y}|$, $\forall \mathbf{x},\mathbf{y} \in G$. Define $k_0^{(n)}(\mathbf{x},\mathbf{y})$ by $k_0^{(1)}(\mathbf{x},\mathbf{y}) = k_0(\mathbf{x},\mathbf{y})$ and $k_0^{(n)}(\mathbf{x},\mathbf{y}) = \int_G k_0(\mathbf{x},\mathbf{z}) k_0^{(n-1)}(\mathbf{z},\mathbf{y}) d\mathbf{z}$. Then $|k_0^{(2)}(\mathbf{x},\mathbf{y})| \le K_2 \log(1/|\mathbf{x}-\mathbf{y}|) + K_3$ and $|k_0^{(3)}(\mathbf{x},\mathbf{y})| \le K_4$ for some constants K_2, K_3 and K_4. Put $K_0^{(n)} f(\mathbf{x}) = \int_G k_0^{(n)}(\mathbf{x},\mathbf{y}) f(\mathbf{y}) d\mathbf{y}$.

A fundamental solution $\Gamma(\mathbf{x},\mathbf{y})$ of L on G can be constructed by

$$\Gamma(\mathbf{x},\mathbf{y}) = \Gamma_0(\mathbf{x},\mathbf{y}) + \int_G \Gamma_0(\mathbf{x},\mathbf{z}) \Phi(\mathbf{z},\mathbf{y}) d\mathbf{z} + \sum \alpha_i(\mathbf{x}) \beta_i(\mathbf{y}) \tag{7.15}$$

for suitable continuous functions $\Phi(\mathbf{x},\mathbf{y})$, $\alpha_i(\mathbf{x})$ and $\beta_i(\mathbf{y})$. In order to make Γ to satisfy $-L\Gamma(\mathbf{x},\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y})$, $\Phi(\mathbf{x},\mathbf{y})$ needs to be a solution of the following Fredholm integral equation.

$$\Phi(\mathbf{x},\mathbf{y}) = k_0(\mathbf{x},\mathbf{y}) + \int_G k_0(\mathbf{x},\mathbf{z}) \Phi(\mathbf{z},\mathbf{y}) d\mathbf{z} + \sum L\alpha_i(\mathbf{x}) \beta_i(\mathbf{y}). \tag{7.16}$$

Note that $k_0^{(n)}(\mathbf{x},\mathbf{y})$ is continuous on G for any $n \ge 3$. Let us take a continuous function

$$g(\mathbf{x},\mathbf{y}) = k_0^{(4)}(\mathbf{x},\mathbf{y}) + k_0^{(5)}(\mathbf{x},\mathbf{y}) + k_0^{(6)}(\mathbf{x},\mathbf{y}) + \sum (K_0^{(3)} + K_0^{(4)} + K_0^{(5)})(L\alpha_i)(\mathbf{x}) \beta_i(\mathbf{y}).$$

Here $\alpha_i = \beta_i = 0$ for all i if $\lambda = 1$ is not an eigenvalue of the dual operator $(K_0^*)^{(3)}$ on $C_b(G)$ of $K_0^{(3)}$ defined by $(K_0^*)^{(3)} f(\mathbf{x}) = \int (k_0^*)^{(3)}(\mathbf{x},\mathbf{y}) f(\mathbf{y}) d\mathbf{y}$ with $k_0^*(\mathbf{x},\mathbf{y}) = k_0(\mathbf{y},\mathbf{x})$, while, if $\lambda = 1$ is an eigenvalue, then α_i, β_i are chosen to satisfy $(g(\cdot,\mathbf{y}), \psi_j) = 0$ for all eigenfunctions $\{\psi_j\}$ corresponding to the eigenvalue $\lambda = 1$ of $(K_0^*)^{(3)}$. Then the Fredholm equation $w(\mathbf{x},\mathbf{y}) = K_0^{(3)} w(\mathbf{x},\mathbf{y}) + g(\mathbf{x},\mathbf{y})$ has a unique continuous solution $w(\mathbf{x},\mathbf{y})$ for any $\mathbf{y} \in G$. Using this solution, the unique solution of (7.16) is given by $\Phi(\mathbf{x},\mathbf{y}) = k_0(\mathbf{x},\mathbf{y}) + k_0^{(2)}(\mathbf{x},\mathbf{y}) + k_0^{(3)}(\mathbf{x},\mathbf{y}) + w(\mathbf{x},\mathbf{y})$. We notice that, according to the construction of Γ from Γ_0 by (7.15),

$$\Gamma(\mathbf{y},\mathbf{z}) - \Gamma_0(\mathbf{y},\mathbf{z}) \text{ is bounded in } (\mathbf{y},\mathbf{z}) \in G \times G. \tag{7.17}$$

We now proceed to a proof of (4.19) with $\kappa = \Lambda/\lambda$. For $\mathbf{x} \in B(S-1)$ and $0 < 5r \le t \le 1/3$, let $\mu^{\mathbf{x},r}$ be the equilibrium measure for $\overline{B(\mathbf{x},r)}$ relative to the admissible set $F = \overline{B(S+1)} \setminus B(S)$ for the Dirichlet form $\mathbf{a}$ on $H^1(\mathbb{C})$. We first show that the logarithmic potential

$$U\mu^{\mathbf{x},r}(\mathbf{y}) = \frac{1}{\pi} \int \log \frac{1}{|\mathbf{y}-\mathbf{z}|} \mu^{\mathbf{x},r}(d\mathbf{z}), \ \mathbf{y} \in \mathbb{C},$$

of $\mu^{\mathbf{x},r}$ has the properties

$$\langle \mu^{\mathbf{x},r}, U\mu^{\mathbf{x},r} \rangle < \infty \quad \text{and} \quad U\mu^{\mathbf{x},r} \in L^2_{\mathrm{loc}}(\mathbb{C}). \tag{7.18}$$

Since $\mu^{\mathbf{x},r}$ is a measure of 0-order finite energy for the perturbed form $\mathbf{a}^g$ of $\mathbf{a}$ by $g = 1_F$, so it is for the perturbed Dirichlet integral $(1/2)\mathbf{D}(u,u) + (u,u)_g$.

Denote by $\acute{\mathbb{M}}$ the planar Brownian motion. $\acute{R}^g(\mathbf{x},\mathbf{y})$ and $\acute{R}^{\mathbb{C}\setminus F}(\mathbf{x},\mathbf{y})$ denote the 0-order resolvent density of the subproces of $\acute{\mathbb{M}}$ by $\exp[-\int_0^t I_F(X_s)ds]$ and that of the part of $\acute{\mathbb{M}}$ on the set $\mathbb{C}\setminus F$, respectively. Then $\acute{R}^{\mathbb{C}\setminus F}(\mathbf{x},\mathbf{y}) \le \acute{R}^g(\mathbf{x},\mathbf{y})$ so that

$$\langle \mu^{\mathbf{x},r}, \acute{R}^{\mathbb{C}\setminus F}\mu^{\mathbf{x},r}\rangle \le \langle \mu^{\mathbf{x},r}, \acute{R}^g\mu^{\mathbf{x},r}\rangle < \infty, \text{ and } \acute{R}^{\mathbb{C}\setminus F}\mu^{\mathbf{x},r} \in H^1_{0,e}(\mathbb{C}\setminus F) \subset \mathrm{BL}(\mathbb{C}) \subset L^2_{\mathrm{loc}}(\mathbb{C}).$$

According to the fundamental identity of the logarithmic potential (cf. [F2, (2.13)]),

$$U\mu^{\mathbf{x},r}(\mathbf{y}) = \acute{R}^{\mathbb{C}\setminus F}\mu^{\mathbf{x},r}(\mathbf{y}) + \acute{H}_F U\mu^{\mathbf{x},r}(\mathbf{y}) - W_F(\mathbf{y}), \quad \mathbf{y}\in\mathbb{C},$$

which readily implies (7.18).

Define $\Gamma\mu^{\mathbf{x},r}(\mathbf{y}) = \int \Gamma(\mathbf{y},\mathbf{z})\mu^{\mathbf{x},r}(d\mathbf{z})$, $y\in\mathbb{C}$. $\Gamma_0\mu^{\mathbf{x},r}$ is defined similarly. Since $\Gamma_0(\mathbf{x},\mathbf{y})$ is bounded by $K_5\log(1/|\mathbf{x}-\mathbf{y}|) + K_6$ for some constants K_5 and K_6, we have $\Gamma_0\mu^{\mathbf{x},r}\in L^2_{\mathrm{loc}}(\mathbb{C})$ by (7.18). By (7.17), this also holds for Γ in place of Γ_0.

Put $A^{-1}(\mathbf{y}) = (a^{ij}(\mathbf{y}))$. Since the weak derivative $\nabla\Gamma_0\mu^{\mathbf{x},r}$ is given by

$$\nabla\Gamma_0\mu^{\mathbf{x},r}(\mathbf{w}) = \int_G \frac{1}{\pi(\det(A^{-1}(\mathbf{y})))^{1/2}} \frac{A^{-1}(\mathbf{y})(\mathbf{w}-\mathbf{y})}{{}^t(\mathbf{w}-\mathbf{y})A^{-1}(\mathbf{y})(\mathbf{w}-\mathbf{y})}\mu^{\mathbf{x},r}(d\mathbf{y}),$$

we get

$$\begin{aligned}\int_G |\nabla\Gamma_0\mu^{\mathbf{x},r}(\mathbf{w})|^2 d\mathbf{w} &\le K_7 \int\int\int \frac{1}{|\mathbf{w}-\mathbf{y}|\,|\mathbf{w}-\mathbf{z}|} d\mathbf{w}\mu^{\mathbf{x},r}(d\mathbf{y})\mu^{\mathbf{x},r}(d\mathbf{z}) \\ &\le K_8\int\int \log\frac{1}{|\mathbf{y}-\mathbf{z}|}\,\mu^{\mathbf{x},r}(d\mathbf{y})\mu^{\mathbf{x},r}(d\mathbf{z}) + K_9.\end{aligned}$$

which is finite by (7.18). Consequently $\Gamma_0\mu^{\mathbf{x},r}\in\mathrm{BL}(G)$. By (7.15), $\Gamma\mu^{\mathbf{x},r}$ also belongs to the space $\mathrm{BL}(G)$. Since the disk G is an extendable domain for BL-functions ([J]), there exists $\Psi\in\mathrm{BL}(\mathbb{C})$ such that $\Psi\big|_G = \Gamma\mu^{\mathbf{x},r}$.

In what follows, we let $T = S - 1/4$. By virtue of Lemma 3.8, it holds that

$$\widehat{R}\mu^{\mathbf{x},r} - H_{\mathbb{C}\setminus B(T)}\widehat{R}\mu^{\mathbf{x},r} = R^{B(T)}\mu^{\mathbf{x},r} \quad \text{q.e.}$$

Further, if we let $\mathcal{F}_{e,B(T)} = \{u\in\mathrm{BL}(\mathbb{C}) : \widetilde{u} = 0 \text{ q.e. on } \mathbb{C}\setminus B(T)\}$, then

$$R^{B(T)}\mu^{\mathbf{x},r}\in\mathcal{F}_{e,B(T)}, \text{ and } \mathbf{a}(R^{B(T)}\mu^{\mathbf{x},r}, v) = \langle\mu^{\mathbf{x},r},\widetilde{v}\rangle,\ \forall v\in\mathcal{F}_{e,B(T)}.$$

Define $\Psi_{B(T)}(\mathbf{y}) = \Psi(\mathbf{y}) - H_{\mathbb{C}\setminus B(T)}\Psi(\mathbf{y})$, $\mathbf{y}\in\mathbb{C}$. As $\Psi\in\mathrm{BL}(\mathbb{C})$, $\Psi_{B(T)}\in\mathcal{F}_{e,B(T)}$ and $H_{\mathbb{C}\setminus B(T)}\Psi$ is $\mathbf{a}$-harmonic on $B(T)$, namely, $\mathbf{a}(H_{\mathbb{C}\setminus B(T)}\Psi, v) = 0$, $\forall v\in\mathcal{F}_{e,B(T)}$. Since Ψ equals $\Gamma\mu^{\mathbf{x},r}$ on G and Γ is a fundamental solution of L on G, we have $\mathbf{a}(\Psi_{B(T)}, v) = \langle\mu^{\mathbf{x},r}, v\rangle$, $\forall v\in\mathcal{F}_{e,B(T)}\cap C_c(B(T))$. Therefore $\mathbf{a}(R^{B(T)}\mu^{\mathbf{x},r} - \Psi_{B(T)}, R^{B(T)}\mu^{\mathbf{x},r} - \Psi_{B(T)}) = 0$, which in turn implies

$$\widehat{R}\mu^{\mathbf{x},r}(\mathbf{y}) - \Gamma\mu^{\mathbf{x},r}(\mathbf{y}) = H_{\partial B(T)}\widehat{R}\mu^{\mathbf{x},r}(\mathbf{y}) - H_{\partial B(T)}\Gamma\mu^{\mathbf{x},r}(\mathbf{y}), \text{ for a.e. } \mathbf{y}\in B(T)\setminus\overline{B}(\mathbf{x},r), \qquad (7.19)$$

where $\widehat{R}\mu^{\mathbf{x},r}$ is a version of $R\mu^{\mathbf{x},r}$ introduced in Lemma 4.4.

By Lemma 4.6,

$$\sup_{\mathbf{y}\in B(T)}\ \sup_{\mathbf{x}\in B(S-1),0<r<1/8} |H_{\partial B(T)}\widehat{R}\mu^{\mathbf{x},r}(\mathbf{y})| \le \sup_{\mathbf{z}\in\partial B(T)}\ \sup_{\mathbf{x}\in B(S-1),0<r<1/8} |\widehat{R}\mu^{\mathbf{x},r}(\mathbf{z})| =: \ell_1 < \infty.$$

By (7.15), $\Gamma(\mathbf{y},\mathbf{z})$ is jointly continuous on $G\times G$ off the diagonal set, and consequently

$$\sup_{\mathbf{y}\in B(T)}\ \sup_{\mathbf{x}\in B(S-1),\,0<r<1/8}|H_{\partial B(T)}\Gamma\mu^{\mathbf{x},r}(\mathbf{y})|\le\sup_{\mathbf{y}\in\partial B(T),\,z\in B(S-1/2)}|\Gamma(\mathbf{y},\mathbf{z})|=:\ell_2<\infty.$$

Therefore, it follows from (7.19) and Lemma 4.4 that

$$\sup_{\mathbf{x}\in B(S-1),\,0<r<1/8}\ \sup_{\mathbf{y}\in B(T)\setminus B(\mathbf{x},r)}|\widehat{R}\mu^{\mathbf{x},r}(\mathbf{y})-\Gamma\mu^{\mathbf{x},r}(\mathbf{y})|\le\ell_1+\ell_2<\infty.$$

By taking (7.17) into account, it holds further that $\Gamma\mu^{\mathbf{x},r}(\mathbf{y})-\Gamma_0\mu^{\mathbf{x},r}(\mathbf{y})$ is bounded uniformly in $\mathbf{x}\in B(S-1)$ and $0<r<1/2$.

Hence, there exists a constant K_9 such that, for $\mathbf{x}\in B(S-1)$ and $0<4r\le t<1/2$,

$$\begin{aligned}&\max\left\{\widehat{R}\mu^{\mathbf{x},r}(\mathbf{y}):\mathbf{y}\in\partial B(\mathbf{x},t)\right\}\le\max\left\{\Gamma_0\mu^{\mathbf{x},r}(\mathbf{y}):\mathbf{y}\in\partial B(\mathbf{x},t)\right\}+K_9\\&\le\frac{\Lambda}{\pi}\max\left\{\int_{B(\mathbf{x},r)}\log\frac{\Lambda}{|\mathbf{y}-\mathbf{z}|^2}\mu^{\mathbf{x},r}(d\mathbf{z}):\mathbf{y}\in\partial B(\mathbf{x},t)\right\}+K_9.\end{aligned}$$

Since $(3/4)t\le|\mathbf{y}-\mathbf{z}|\le(5/4)t$ for any $\mathbf{y}\in\partial B(\mathbf{x},t)$ and $\mathbf{z}\in\partial B(\mathbf{x},r)$, the last expression in the above display is dominated by

$$\begin{aligned}&\frac{\Lambda}{\pi}\min\left\{\int_{B(\mathbf{x},r)}\log\frac{25\Lambda}{9|\mathbf{y}-\mathbf{z}|^2}\mu^{\mathbf{x},r}(d\mathbf{z}):\mathbf{y}\in\partial B(\mathbf{x},t)\right\}+K_9.\\&\le\frac{\Lambda}{\lambda}\min\left\{\frac{\lambda}{\pi}\int_{B(\mathbf{x},r)}\log\frac{\lambda}{|\mathbf{y}-\mathbf{z}|^2}\mu^{\mathbf{x},r}(d\mathbf{z}):\mathbf{y}\in\partial B(\mathbf{x},t)\right\}+\frac{\Lambda}{\pi}(\log(25\Lambda/9)-\log\lambda)+K_9\\&\le\frac{\Lambda}{\lambda}\min\{\Gamma_0\mu^{\mathbf{x},r}(\mathbf{y}):\mathbf{y}\in\partial B(\mathbf{x},t)\}+K_{10}\\&\le\frac{\Lambda}{\lambda}\min\{\widehat{R}\mu^{\mathbf{x},r}(\mathbf{y}):\mathbf{y}\in\partial B(\mathbf{x},t)\}+K_9+K_{10}\end{aligned}$$

for $K_{10}=K_9+(\Lambda/\pi)(\log(25\Lambda/9)-\log\lambda)$. Therefore (4.19) holds for $\kappa=\Lambda/\lambda$ and $C_2=K_9+K_{10}$.

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